

THE EFFECT OF THE HARDY POTENTIAL IN SOME CALDERÓN-ZYGMUND PROPERTIES FOR THE FRACTIONAL LAPLACIAN

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ABSTRACT. The goal of this paper is to study the effect of the Hardy potential on the existence and summability of solutions to a class of nonlocal elliptic problems

$$\begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where $(-\Delta)^s$, $s \in (0, 1)$, is the fractional laplacian operator, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary such that $0 \in \Omega$ and $N > 2s$. We will mainly consider the solvability in two cases:

- (1) The linear problem, that is, $f(x, t) = f(x)$, where according to the summability of the datum f and the parameter λ we give the summability of the solution u .
- (2) The problem with a nonlinear term $f(x, t) = \frac{h(x)}{t^\sigma}$ for $t > 0$. In this case, existence and regularity will depend on the value of σ and on the summability of h .

Looking for optimal results we will need a weak Harnack inequality for elliptic operators with *singular coefficients* that seems to be new.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

This work deals with the following problem

$$(1) \quad \begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0, 1)$ is such that $2s < N$, $\Omega \subset \mathbb{R}^N$ is a bounded regular domain containing the origin and f is a measurable function satisfying suitable hypotheses.

Recall that we define the fractional Laplacian $(-\Delta)^s$ as the operator given by the Fourier multiplier $|\xi|^{2s}$. That is, for $u \in \mathcal{S}(\mathbb{R}^N)$,

$$\mathcal{F}((-\Delta)^s u)(\xi) := |\xi|^{2s} \mathcal{F}(u)(\xi).$$

A computation involving the inverse Fourier transform of a homogeneous tempered distribution gives the formal expression of the fractional Laplacian as an integral operator, see for instance [33].

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More precisely, if $u \in \mathcal{S}(\mathbb{R}^N)$,

$$(2) \quad (-\Delta)^s u(x) := a_{N,s} \text{ P.V. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad s \in (0, 1),$$

where

$$(3) \quad a_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\xi_1)}{|\xi|^{N+2s}} d\xi \right)^{-1} = 2^{2s-1} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}.$$

Due to its second term, problem (1) is related to the following Hardy inequality, proved in [24] (see also [12, 23, 34, 36]),

$$(4) \quad \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}^2 d\xi \geq \Lambda_{N,s} \int_{\mathbb{R}^N} |x|^{-2s} u^2 dx \quad \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^N),$$

where

$$(5) \quad \Lambda_{N,s} := 2^{2s} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$$

is optimal and it is not attained. Moreover,

$$\lim_{s \rightarrow 1} \Lambda_{N,s} = \left(\frac{N-2}{2} \right)^2,$$

the classical Hardy constant.

The Hardy inequality (4) plays an important role, for instance, in a general proof of the *stability of the relativistic matter*, see [23]. We can also rewrite inequality (4) in the form

$$\frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \geq \Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx, \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^N),$$

which we will often use along the paper. As we will see, this critical value $\Lambda_{N,s}$ will also play a fundamental role concerning solvability. In particular, we already know that for $\lambda > \Lambda_{N,s}$ problem (1) has no positive supersolution (see for example [11, 20]). Hence, from now on we will assume $0 < \lambda \leq \Lambda_{N,s}$.

If $f(x, s) = f(x)$, problem (1) is reduced to the linear case

$$(6) \quad \begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and our first goal will be to obtain the optimal summability of u according to the summability of the datum f and the parameter λ . The case $\lambda = 0$ can be found in [28], where some Calderón-Zygmund type results are obtained.

Since this problem is linear, we will assume, without loss of generality, that the datum f is positive and we will deal with positive solutions.

The influence of the Hardy potential in the local case ($s = 1$) was studied in [14]. The main results there can be summarized as follows. Suppose $f \in L^m(\Omega)$ and let

$$\lambda(m) := \frac{N(m-1)(N-2m)}{m^2}.$$

Then if $0 < \lambda < \lambda(m)$ the solution to problem (6) verifies the same Calderón-Zygmund inequalities as for $\lambda = 0$. On the contrary, if $\lambda \geq \lambda(m)$ it is possible to find counterexamples of these results, so the regularity does not hold (see [14] for details).

In particular, if $m > \frac{N}{2}$ the solutions are unbounded, and if $m = 1$ there is no solution in general. Indeed, the necessary and sufficient condition in order to have solvability is to assume the following integrability of the datum with respect to the weight,

$$\int_{\Omega} f|x|^{-\alpha_1} dx < \infty,$$

where $\alpha_1 := \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}$ (see [7]).

In order to find an analogous optimal condition for problem (6), we will need a *weak Harnack inequality* for a singular weighted nonlocal operator (that we will define in (10)). This study, performed in Section 3, requires the combination of techniques on elliptic operators and very involved computations on nonlocal radial integrals, and it provides the precise behavior of the solutions around the origin. This result will be the key in the proofs of existence and regularity in the next Sections.

As an application and as a complement to the results in [11], we will also study a semilinear problem which is singular at the boundary. More precisely, we will consider the problem

$$(7) \quad \begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = \frac{h(x)}{u^\sigma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

The local case ($s = 1$) with $\lambda = 0$ was studied in [13]. Here the authors proved that for all $h \in L^1(\Omega)$, there exists at least one distributional solution. Regularity is obtained according to the regularity of h and the value of σ .

The main purpose of this work is to obtain the same kind of results for the fractional Laplacian framework, whose nonlocal behavior introduces new difficulties. Some partial results have been already obtained in [1], including the p -Laplacian like operator.

The paper is organized as follows. In Section 2 we precise the meaning of solutions that will be used along the work, with the corresponding functional setting. Some useful tools as the Picone inequality, compactness results and certain algebraic inequalities are also proved here.

In Section 3 we prove a weighted singular version of the Harnack inequality. Notice that, using the *ground state transformation* stated in Lemma 2.8, the weak Harnack inequality gives the exact blow up rate for the positive supersolutions to (1) near the origin. As we said, this theorem will be the key for the optimality in the results of the following sections.

In Section 4 we treat the linear problem (6). According to λ and the summability of f , we find the optimal summability of the solution u for certain values of the spectral parameter λ . In particular, we see that the local techniques applied in [14] do not give complete information in this framework, leaving the optimality for certain ranges of λ as an open problem. We analyze this situation in detail in this section.

Finally, last section is devoted to study problem (7). We prove existence and regularity results depending on the value of σ .

2. FUNCTIONAL SETTING AND USEFUL TOOLS

Let $s \in (0, 1)$. For any $p \in [1, \infty)$ and $\Omega \subseteq \mathbb{R}^N$, we define $W^{s,p}(\Omega)$ as follows,

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \text{ s.t. } \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} \in L^p(\Omega \times \Omega) \right\}.$$

We focus on the case $p = 2$, where the fractional Sobolev spaces $H^s(\Omega) := W^{s,2}(\Omega)$ turn out to be Hilbert spaces. Moreover, if $\Omega = \mathbb{R}^N$, the Fourier transform provides an alternative definition.

Definition 2.1. For $0 < s < 1$, we define the fractional Sobolev space of order s as

$$H^s(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) \text{ s.t. } |\xi|^s \mathcal{F}(u)(\xi) \in L^2(\mathbb{R}^N)\}.$$

Hence by Plancherel identity, we obtain a new expression for the norm of the Hilbert space $H^s(\mathbb{R}^N)$ (see [23] for a detailed proof).

Proposition 2.2. Let $N \geq 1$ and $0 < s < 1$. Then for all $u \in H^s(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)(\xi)|^2 d\xi = \frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy,$$

where $a_{N,s}$ is the constant defined in (3).

Moreover we can extend by density the operator $(-\Delta)^s u$ defined in (2) from $\mathcal{S}(\mathbb{R}^N)$ to $H^s(\mathbb{R}^N)$. In this way, the associated scalar product can be reformulated as follows

$$\begin{aligned} \langle u, v \rangle_{H^s(\mathbb{R}^N)} &:= \langle (-\Delta)^s u, v \rangle + (u, v) \\ &:= P.V. \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} uv dx. \end{aligned}$$

We call $\|\cdot\|_{H_0^s(\mathbb{R}^N)}$ the induced norm by this scalar product. Summarizing the previous result we obtain the following useful formulation, that includes the corresponding *integration by parts* (see for instance [18]).

Proposition 2.3. Let $s \in (0, 1)$ and $u \in H^s(\mathbb{R}^N)$. Then,

$$\frac{a_{N,s}}{2} \langle (-\Delta)^s u, u \rangle = \frac{a_{N,s}}{2} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}^2 = \| |\xi|^s \mathcal{F}u \|_{L^2(\mathbb{R}^N)}^2.$$

The dual space of $H^s(\mathbb{R}^N)$ is defined by

$$H^{-s}(\mathbb{R}^N) = \{f \in \mathcal{S}'(\mathbb{R}^N) / |\xi|^{-s} \mathcal{F}(f) \in L^2(\mathbb{R}^N)\}.$$

The following properties are immediate:

- (1) $(-\Delta)^s : H^s(\mathbb{R}^N) \rightarrow H^{-s}(\mathbb{R}^N)$ is a continuous operator.
- (2) $(-\Delta)^s$ is a symmetric operator in $H^s(\mathbb{R}^N)$, that is,

$$\langle (-\Delta)^s u, v \rangle = \langle u, (-\Delta)^s v \rangle, \quad u, v \in H^s(\mathbb{R}^N).$$

- (3) Denoting also by $\langle \cdot, \cdot \rangle$ the natural duality product between $H^s(\mathbb{R}^N)$ and $H^{-s}(\mathbb{R}^N)$, then

$$|\langle (-\Delta)^s u, v \rangle| \leq \|u\|_{H^s(\mathbb{R}^N)} \|v\|_{H^s(\mathbb{R}^N)}.$$

We define now the space $H_0^s(\Omega)$ as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm of $H^s(\mathbb{R}^N)$. Notice that if $u \in H_0^s(\Omega)$, we have $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ and we can write

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy = \iint_{D_\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy$$

where

$$D_\Omega := \mathbb{R}^N \times \mathbb{R}^N \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega).$$

Remark 2.4. If Ω is a bounded domain and $u \in C_0^\infty(\Omega)$, then by setting

$$|||u|||_{H_0^s(\Omega)} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2},$$

and using a Poincaré type inequality (see [4] or [18]), we can prove that $||| \cdot |||_{H_0^s(\Omega)}$ and $\| \cdot \|_{H_0^s(\Omega)}$ are equivalent norms.

If we denote by $H^{-s}(\Omega) := [H_0^s(\Omega)]^*$ the dual space of $H_0^s(\Omega)$, then

$$(-\Delta)^s : H_0^s(\Omega) \rightarrow H^{-s}(\Omega),$$

is a continuous operator.

We give the meaning of solutions that will be used along the paper: *i)* *energy solutions* when the variational framework can be used and *ii)* *weak solutions* for data that are integrable but not in the dual space.

Recall that we assume $0 \in \Omega$.

Definition 2.5. Assume $0 < \lambda < \Lambda_{N,s}$. For $f \in H^{-s}(\Omega)$ we say that $u \in H_0^s(\Omega)$ is a *finite energy solution* to (1) if

$$\frac{a_{N,s}}{2} \langle (-\Delta)^s u, w \rangle - \lambda \int_{\Omega} \frac{uw}{|x|^{2s}} dx = \langle f, w \rangle, \quad \forall w \in H_0^s(\Omega).$$

If $\lambda < \Lambda_{N,s}$, then existence and uniqueness of a solution $u \in H_0^s(\Omega)$ for all $f \in H^{-s}(\Omega)$ easily follows.

Remark 2.6. If $\lambda = \Lambda_{N,s}$, the same result holds but in a space $H(\Omega)$ defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|\phi\|_{H(\Omega)}^2 := \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy - \Lambda_{N,s} \int_{\Omega} \frac{\phi^2}{|x|^{2s}} dx.$$

By using the improved Hardy inequality (see for instance [8]) we get that $H(\Omega)$ is a Hilbert space and $H_0^s(\Omega) \subsetneq H(\Omega) \subsetneq W_0^{s,q}(\Omega)$, for all $q < 2$.

To deal with the case of a general $f \in L^1(\Omega)$, we need to define the notion of *weak solution*, where we only request the regularity needed to give weak sense to the equation.

Since the operator is nonlocal, we need to precise the class of test function to be considered, that precisely is,

$$(8) \quad \mathcal{T} := \{ \phi : \mathbb{R}^N \rightarrow \mathbb{R} \mid (-\Delta)^s \phi = \varphi, \varphi \in L^\infty(\Omega) \cap C^\alpha(\Omega), 0 < \alpha < 1, \phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega \}.$$

Notice that every $\phi \in \mathcal{T}$ belongs in particular to $L^\infty(\Omega)$ (see [28]) and moreover it is a strong solution to the equation $(-\Delta)^s \phi = \varphi$. See for instance [30] and [31].

Definition 2.7. Assume $f \in L^1(\Omega)$. We say that $u \in L^1(\Omega)$ is a weak supersolution (subsolution) of problem (1) if $\frac{u}{|x|^{2s}} \in L^1(\Omega)$, $u = 0$ in $\mathbb{R}^N \setminus \Omega$, and for all nonnegative $\phi \in \mathcal{T}$, the following inequality holds,

$$\int_{\Omega} u (-\Delta)^s \phi dx - \lambda \int_{\Omega} \frac{u \phi}{|x|^{2s}} dx \geq (\leq) \int_{\Omega} f \phi dx.$$

If u is super and subsolution, then we say that u is a weak solution.

Notice that, if $u \in C_0^\infty(\mathbb{R}^N)$, Frank, Lieb and Seiringer proved in [23] the following result.

Lemma 2.8. (*Ground State Representation*) Let $0 < \gamma < \frac{N-2s}{2}$. If $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ and $v(x) := |x|^\gamma u(x)$, then

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi - (\Lambda_{N,s} + \Phi_{N,s}(\gamma)) \int_{\mathbb{R}^N} |x|^{-2s} |u(x)|^2 dx = a_{N,s} \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} \frac{dx}{|x|^\gamma} \frac{dy}{|y|^\gamma},$$

where

$$\Phi_{N,s}(\gamma) = 2^{2s} \left(\frac{\Gamma(\frac{\gamma+2s}{2}) \Gamma(\frac{N-\gamma}{2})}{\Gamma(\frac{N-\gamma-2s}{2}) \Gamma(\frac{\gamma}{2})} - \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})} \right).$$

Notice that in particular this representation proves that the constant $\Lambda_{N,s}$ is optimal and is not attained. See [23, Remark 4.2] for details.

Using this representation with $\lambda = \Lambda_{N,s} + \Phi_{N,s}(\gamma)$, we obtain that if u solves

$$\begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

then v satisfies

$$(9) \quad \begin{cases} L_\gamma v = |x|^{-\gamma} f(x, |x|^{-\gamma} v) =: g(x, v) & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

with

$$(10) \quad L_\gamma v := a_{N,s} \text{ P.V. } \iint_{\mathbb{R}^N} \frac{v(x) - v(y)}{|x - y|^{N+2s}} \frac{dy}{|x|^\gamma |y|^\gamma}.$$

Observe that if $\gamma \rightarrow 0$, then $\Phi_{N,s}(\gamma) \rightarrow -\Lambda_{N,s}$ and $\lambda \rightarrow 0$. On the other hand, if $\gamma \rightarrow \frac{N-2s}{2}$, then $\Phi_{N,s}(\gamma) \rightarrow 0$ and $\lambda \rightarrow \Lambda_{N,s}$.

To analyze the behavior and the regularity of u , we deal with the same questions for v . Thus, we need to work in fractional Sobolev spaces with admissible weights. For simplicity of typing, we denote

$$d\mu := \frac{dx}{|x|^{2\gamma}} \quad \text{and} \quad d\nu := \frac{dxdy}{|x - y|^{N+2s} |x|^\gamma |y|^\gamma}.$$

For $\Omega \subseteq \mathbb{R}^N$, we define the weighted fractional Sobolev space $Y^{s,\gamma}(\Omega)$ as follows

$$Y^{s,\gamma}(\Omega) := \left\{ \phi \in L^2(\Omega, d\mu) \text{ s.t. } \int_\Omega \int_\Omega (\phi(x) - \phi(y))^2 d\nu < +\infty \right\}.$$

It is clear that $Y^{s,\gamma}(\Omega)$ is a Hilbert space endowed with the norm

$$\|\phi\|_{Y^{s,\gamma}(\Omega)} := \left(\int_\Omega |\phi(x)|^2 d\mu + \int_\Omega \int_\Omega (\phi(x) - \phi(y))^2 d\nu \right)^{\frac{1}{2}}.$$

The following extension lemma can be proved by using the same arguments of [10] (see also [18]).

Lemma 2.9. Let $\Omega \subset \mathbb{R}^N$ be a smooth domain. Then for all $w \in Y^{s,\gamma}(\Omega)$, there exists $\tilde{w} \in Y^{s,\gamma}(\mathbb{R}^N)$ such that $\tilde{w}|_\Omega = w$ and

$$\|\tilde{w}\|_{Y^{s,\gamma}(\mathbb{R}^N)} \leq C \|w\|_{Y^{s,\gamma}(\Omega)},$$

where $C := C(N, s, \Omega, \gamma) > 0$.

We define the space $Y_0^{s,\gamma}(\Omega)$ as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm of $Y^{s,\gamma}(\Omega)$. If Ω is a bounded regular domain, then as in the case $\gamma = 0$, we have the next Poincaré inequality (a proof can be found in the Appendix B of [4]).

Theorem 2.10. *There exists a positive constant $C := C(\Omega, N, s, \gamma)$ such that for all $\phi \in \mathcal{C}_0^\infty(\Omega)$, we have*

$$C \int_{\Omega} \phi^2(x) d\mu \leq \int_{\Omega} \int_{\Omega} (\phi(x) - \phi(y))^2 d\nu.$$

As a consequence we reach that if Ω is a bounded domain, we can consider $Y_0^{s,\gamma}(\Omega)$ with the equivalent norm

$$|||\phi|||_{Y_0^{s,\gamma}(\Omega)} := \left(\int_{\Omega} \int_{\Omega} (\phi(x) - \phi(y))^2 d\nu \right)^{\frac{1}{2}}.$$

In [3], the authors prove the following weighted Sobolev inequality.

Proposition 2.11. *Consider $0 < s < 1$ such that $N > 2s$ and $0 < \gamma < \frac{N-2s}{2}$. Then, for all $v \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, there exists a positive constant $S = S(N, s, \gamma)$ such that*

$$\frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} \frac{dx}{|x|^\gamma} \frac{dy}{|y|^\gamma} \geq S \left(\int_{\mathbb{R}^N} \frac{|v(x)|^{2_s^*}}{|x|^{2_s^* \gamma}} \right)^{\frac{2}{2_s^*}},$$

where $2_s^* := \frac{2N}{N-2s}$.

If $\Omega \subset \mathbb{R}^N$ is a bounded domain and $\gamma = \frac{N-2s}{2}$, then for all $q < 2$, there exists a positive constant $C = C(\Omega, \gamma, s, q)$ such that

$$\frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} \frac{dx}{|x|^\gamma} \frac{dy}{|y|^\gamma} \geq C \left(\int_{\mathbb{R}^N} \frac{|v(x)|^{2_{s,q}^*}}{|x|^{2_{s,q}^* \gamma}} \right)^{\frac{2}{2_{s,q}^*}},$$

for all $v \in \mathcal{C}_0^\infty(\Omega)$, where $2_{s,q}^* := \frac{2N}{N-qs}$.

Combining the previous proposition and the extension lemma we get the next Sobolev inequality in the space $Y_0^{s,\gamma}(\Omega)$.

Proposition 2.12. *Let Ω be a bounded regular domain and suppose that the hypotheses of Proposition 2.11 hold, then, for all $v \in \mathcal{C}_0^\infty(\Omega)$, there exists a positive constant $S = S(N, s, \gamma, \Omega)$ such that*

$$\frac{a_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} \frac{dx}{|x|^\gamma} \frac{dy}{|y|^\gamma} \geq S \left(\int_{\Omega} \frac{|v(x)|^{2_s^*}}{|x|^{2_s^* \gamma}} \right)^{\frac{2}{2_s^*}}.$$

Proof. Let $v \in \mathcal{C}_0^\infty(\Omega)$ and define \tilde{v} to be the extension of v to \mathbb{R}^N given in Lemma 2.9. Then using Proposition 2.11, we get

$$C \|v\|_{Y^{s,\gamma}(\Omega)} \geq \frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{v}(x) - \tilde{v}(y)|^2}{|x - y|^{N+2s}} \frac{dxdy}{|x|^\gamma |y|^\gamma} \geq S(N, s, \gamma) \left(\int_{\mathbb{R}^N} \frac{|\tilde{v}(x)|^{2_s^*}}{|x|^{2_s^* \gamma}} dx \right)^{\frac{2}{2_s^*}}.$$

Since $\tilde{v}|_{\Omega} = v$, then using Theorem 2.10, we reach that

$$\frac{a_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} \frac{dxdy}{|x|^\gamma |y|^\gamma} \geq S(N, s, \gamma, \Omega) \left(\int_{\Omega} \frac{|v(x)|^{2_s^*}}{|x|^{2_s^* \gamma}} dx \right)^{\frac{2}{2_s^*}}$$

and the result follows. \square

We state now a weighted version of the Poincaré-Wirtinger inequality that we will use later (see Appendix B in [4] for a proof).

Theorem 2.13. *Let $r > 0$ and $w \in Y^{s,\gamma}(B_r)$ and assume that ψ is a radial decreasing function such that $\text{supp } \psi \subset B_r$ and $0 \leq \psi \leq 1$. Define*

$$W_\psi := \frac{\int_{B_r} w(x)\psi(x)d\mu}{\int_{B_r} \psi(x)d\mu}.$$

Then,

$$\int_{B_r} (w(x) - W_\psi)^2 \psi(x) d\mu \leq Cr^{2s} \int_{B_r} \int_{B_r} (w(x) - w(y))^2 \min\{\psi(x), \psi(y)\} d\nu.$$

Finally, we define

$$Y_{loc}^{s,\gamma}(\Omega) := \{u \in L_{loc}^2(\Omega, d\mu) \text{ s.t. } \forall \Omega_1 \subset\subset \Omega, \int_{\Omega_1} \int_{\Omega_1} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \frac{dx}{|x|^\gamma} \frac{dy}{|y|^\gamma} + \int_{\Omega_1} u^2 d\mu < +\infty\}.$$

Remark 2.14. This definition for $\gamma = 0$ is similar. In such a case we will denote the associated space as $H_{loc}^s(\Omega)$.

We consider the following natural definition.

Definition 2.15. Let $v \in Y_{loc}^{s,\gamma}(\Omega)$. We say that v is a supersolution to problem (9) if

$$(11) \quad \iint_{D_{\Omega_1}} (v(x) - v(y))(\varphi(x) - \varphi(y)) d\nu \geq \int_{\Omega_1} g\varphi dx$$

for every nonnegative $\varphi \in Y_0^{s,\gamma}(\Omega_1)$ and every $\Omega_1 \subset\subset \Omega$.

An integral extension involving positive Radon measures of a well-known punctual inequality by Picone (see [29]) was obtained in [6] in the local framework. An extension to the fractional setting has been obtained in [28]. A similar inequality holds for the operator

$$(12) \quad L_{\gamma,\Omega}(w)(x) := a_{N,s} \text{P.V.} \int_{\Omega} \frac{w(x) - w(y)}{|x - y|^{N+2s}} \frac{dy}{|x|^\gamma |y|^\gamma}.$$

Notice that, if $\Omega = \mathbb{R}^N$, L_{γ,\mathbb{R}^N} coincides with L_γ defined in (10).

Theorem 2.16. (*Picone's type Inequality*). *Let $w \in Y^{s,\gamma}(\Omega)$ be such that $w > 0$ in Ω , and assume that $L_{\gamma,\Omega}(w) = \nu$ with $\nu \in L_{loc}^1(\mathbb{R}^N)$ and $\nu \geq 0$. Then for all $v \in C_0^\infty(\Omega)$ we have*

$$(13) \quad \frac{a_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} \frac{dx dy}{|x|^\gamma |y|^\gamma} \geq \langle L_{\gamma,\Omega}(w), \frac{v^2}{w} \rangle$$

where $L_{\gamma,\Omega}w$ is defined by (12).

The proof is the same as in [28], where is based in a punctual inequality. As a consequence, we have the next comparison principle that extends to the weighted fractional framework the classical one obtained by Brezis and Kamin in [15].

Lemma 2.17. *Let Ω be a bounded domain and let f be a nonnegative continuous function such that $f(x, \sigma) > 0$ if $\sigma > 0$ and $\frac{f(x, \sigma)}{\sigma}$ is decreasing. Let $u, v \in Y_0^{s,\gamma}(\Omega)$ be such that $u, v > 0$ in Ω and*

$$\begin{cases} L_{\gamma,\Omega}(u) & \geq f(x, u) \text{ in } \Omega, \\ L_{\gamma,\Omega}(v) & \leq f(x, v) \text{ in } \Omega. \end{cases}$$

Then, $u \geq v$ in Ω .

Proof. The proof is the same as in the case of constant coefficients, since it relies on several pointwise inequalities (see [28] for details). \square

In the sequel we will need the next compactness result.

Lemma 2.18. *Let $\{u_n\}_{n \in \mathbb{N}} \subset Y_0^{s,\gamma}(\Omega)$ be an increasing sequence of nonnegative functions such that $L_{\gamma,\Omega}(u_n) \geq 0$. Assume that $u_n \rightharpoonup u$ weakly in $Y_0^{s,\gamma}(\Omega)$. Then, $u_n \rightarrow u$ strongly in $Y_0^{s,\gamma}(\Omega)$.*

Proof. Since $L_{\gamma,\Omega}(u_n) \geq 0$, then $\langle L_{\gamma,\Omega}(u_n), w_n \rangle \leq 0$, where $w_n := u_n - u$. Thus

$$\int_{\Omega} \int_{\Omega} \frac{(u_n(x) - u_n(y))(w_n(x) - w_n(y))}{|x - y|^{N+2s}} \frac{dx dy}{|x|^{\gamma}|y|^{\gamma}} \leq 0.$$

Since

$$(u_n(x) - u_n(y))(w_n(x) - w_n(y)) = (u_n(x) - u_n(y))^2 - (u_n(x) - u_n(y))(u(x) - u(y))$$

by Young inequality we conclude that

$$\int_{\Omega} \int_{\Omega} \frac{(u_n(x) - u_n(y))^2}{|x - y|^{N+2s}} \frac{dx dy}{|x|^{\gamma}|y|^{\gamma}} \leq \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \frac{dx dy}{|x|^{\gamma}|y|^{\gamma}}.$$

Therefore $\|u_n\|_{Y_0^{s,\gamma}(\Omega)} \leq \|u\|_{Y_0^{s,\gamma}(\Omega)}$ and hence $u_n \rightarrow u$ strongly in $Y_0^{s,\gamma}(\Omega)$. \square

Likewise, we have the following local version of Lemma 2.18.

Lemma 2.19. *Let $\{u_n\}_{n \in \mathbb{N}} \subset Y_0^{s,\gamma}(\Omega)$ be an increasing sequence of nonnegative functions such that $L_{\gamma,\Omega}(u_n) \geq 0$. Assume that $\{u_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $Y_{loc}^{s,\gamma}(\Omega)$, then there exists $u \in Y_{loc}^{s,\gamma}(\Omega)$ such that $u_n \rightarrow u$ strongly in $Y_{loc}^{s,\gamma}(\Omega)$.*

Proof. By using a straightforward modification of Lemma 5.3 in [18], and multiplying the sequence $\{u_n\}_{n \in \mathbb{N}}$ by a Lipschitz cut-off function ψ such that $\psi \equiv 1$ on $\Omega' \subset \Omega$, we can apply Lemma 2.18 to conclude. \square

Remark 2.20. Lemma 2.17, Lemma 2.18 and Lemma 2.19 also hold for $\gamma = 0$, i.e., for the spaces $H_0^s(\Omega)$ and $H_{loc}^s(\Omega)$.

We will also consider nonvariational data, i.e., in $L^1(\Omega)$. In this case, we will use the classical truncating procedure to get *a priori* estimates. Recall that for any $k \geq 0$, $T_k(\sigma)$ and $G_k(\sigma)$, $\sigma \in \mathbb{R}^+$ are defined by

$$(14) \quad T_k(\sigma) := \min\{k, \sigma\} \quad \text{and} \quad G_k(\sigma) := \sigma - T_k(\sigma).$$

Proposition 2.21. *Assume that $v \in H_0^s(\Omega)$:*

- i) *if $\psi \in Lip(\mathbb{R})$ is such that $\psi(0) = 0$, then $\psi(v) \in H_0^s(\Omega)$. In particular, for any $k \geq 0$, $T_k(v), G_k(v) \in H_0^s(\Omega)$;*
- ii) *for any $k \geq 0$,*

$$(15) \quad \|G_k(v)\|_{H_0^s(\Omega)}^2 \leq \int_{\Omega} G_k(v) (-\Delta)^s v \, dx;$$

- iii) *for any $k \geq 0$,*

$$(16) \quad \|T_k(v)\|_{H_0^s(\Omega)}^2 \leq \int_{\Omega} T_k(v) (-\Delta)^s v \, dx.$$

A detailed proof of this result can be seen in [28]. Since the proof relies in a punctual inequality a similar result holds for the *weighted operator*.

The next elementary algebraic inequality will be used in some arguments. See [25] and [2].

For the reader convenience we give a complete proof here.

Lemma 2.22. *Let $s_1, s_2 \geq 0$ and $a > 0$. Then*

$$(17) \quad (s_1 - s_2)(s_1^a - s_2^a) \geq \frac{4a}{(a+1)^2} (s_1^{\frac{a+1}{2}} - s_2^{\frac{a+1}{2}})^2.$$

Proof. Since $\frac{4a}{(a+1)^2} \leq 1$ for $0 \leq a$, if $s_1 = 0$ or $s_2 = 0$ the inequality trivially follows. Hence, we can assume $s_1 > s_2 > 0$. Thus, setting $x := \frac{s_2}{s_1}$, (17) is equivalent to

$$(18) \quad (1-x)(1-x^a) \geq \frac{4a}{(a+1)^2} (1-x^{\frac{a+1}{2}})^2 \text{ for all } x \in (0, 1).$$

We set

$$h(x) := (1-x)(1-x^a)(a+1)^2 - 4a(1-x^{\frac{a+1}{2}})^2,$$

and then we just have to show that $h(x) \geq 0$ for all $x \in (0, 1)$. Moreover, h can be written as

$$h(x) = (a-1)^2(1-x^{\frac{a+1}{2}})^2 - (a+1)^2(x^{\frac{1}{2}} - x^{\frac{a}{2}})^2.$$

First, we assume $a > 1$. We claim that

$$(a-1)(1-x^{\frac{a+1}{2}}) \geq (a+1)(x^{\frac{1}{2}} - x^{\frac{a}{2}}).$$

In fact, let us define

$$h_1(x) := (a-1)(1-x^{\frac{a+1}{2}}) - (a+1)(x^{\frac{1}{2}} - x^{\frac{a}{2}}),$$

so that $h'_1(x) = \frac{(a+1)}{2} \left(-(a-1)x^{\frac{a-1}{2}} - x^{-\frac{1}{2}} + ax^{\frac{a-2}{2}} \right)$.

Using Young inequality, we obtain that

$$x^{\frac{a}{2}-1} \leq \frac{a-1}{a} x^{\frac{a-1}{2}} + \frac{1}{a} x^{-\frac{1}{2}}.$$

Thus $h'_1(x) \leq 0$ and hence $h_1(x) \geq h_1(1) = 0$. Therefore $h(x) \geq 0$ and the result follows in this case.

Consider now the case $a < 1$. We prove the result if we show

$$(1-a)(1-x^{\frac{a+1}{2}}) \geq (a+1)(x^{\frac{a}{2}} - x^{\frac{1}{2}}).$$

Defining

$$h_2(x) := (1-a)(1-x^{\frac{a+1}{2}}) - (a+1)(x^{\frac{a}{2}} - x^{\frac{1}{2}})$$

and using again Young inequality we obtain that $h'_2(x) \leq 0$ for all $x \in (0, 1)$. Thus $h_2(x) \geq h_2(1) = 0$. Hence $h(x) \geq 0$ and we conclude. \square

Finally we state the following classical numerical iteration result proved in [32]) and that we will use later for some boundedness results.

Lemma 2.23. *Let $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a nonincreasing function such that*

$$\psi(h) \leq \frac{M \psi(k)^\delta}{(h-k)^\gamma}, \quad \forall h > k > 0,$$

where $M > 0$, $\delta > 1$ and $\gamma > 0$. Then $\psi(d) = 0$, where $d^\gamma = M \psi(0)^{\delta-1} 2^{\frac{\delta\gamma}{\delta-1}}$.

3. WEAK HARNACK INEQUALITY AND LOCAL BEHAVIOR OF NONNEGATIVE SUPERSOLUTIONS.

Consider the homogeneous equation

$$(-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = 0 \text{ in } \mathbb{R}^N.$$

First, the following result holds (see [11, 20]).

Lemma 3.1. *Let $0 < \lambda \leq \Lambda_{N,s}$. Then $u_{\pm\alpha} := |x|^{-\frac{N-2s}{2} \pm \alpha}$ solves*

$$(-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} \text{ in } \mathbb{R}^N \setminus \{0\},$$

where α is given by the identity

$$(19) \quad \lambda = \lambda(\alpha) = \lambda(-\alpha) = \frac{2^{2s} \Gamma(\frac{N+2s+2\alpha}{4}) \Gamma(\frac{N+2s-2\alpha}{4})}{\Gamma(\frac{N-2s+2\alpha}{4}) \Gamma(\frac{N-2s-2\alpha}{4})}.$$

Moreover $\lambda(\alpha)$ is a positive decreasing continuous function from $\left[0, \frac{N-2s}{2}\right)$ to $(0, \Lambda_{N,s}]$.

Remark 3.2. Notice that $\lambda(\alpha) = \lambda(-\alpha) = m_\alpha m_{-\alpha}$, with $m_\alpha := 2^{\alpha+s} \frac{\Gamma(\frac{N+2s+2\alpha}{4})}{\Gamma(\frac{N-2s-2\alpha}{4})}$.

Denote

$$(20) \quad \gamma := \frac{N-2s}{2} - \alpha \text{ and } \bar{\gamma} := \frac{N-2s}{2} + \alpha,$$

with $0 < \gamma \leq \frac{N-2s}{2} \leq \bar{\gamma} < (N-2s)$. Since $N-2\gamma-2s = 2\alpha > 0$ and $N-2\bar{\gamma}-2s = -2\alpha < 0$, then $|x|^{-\gamma}$ is the unique energy solution of these ones such that $(-\Delta)^{s/2}(|x|^{-\gamma}) \in L_{loc}^2(\mathbb{R}^N)$.

Let u be the energy solution to problem (6) with $0 < \lambda < \Lambda_{N,s}$. By setting $v(x) := |x|^\gamma u(x)$, where γ is defined in (20), it follows that v solves

$$(21) \quad \begin{cases} L_\gamma v(x) &= |x|^{-\gamma} f(x) & \text{in } \Omega, \\ v &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

with $0 < \gamma < \frac{N-2s}{2}$ and L_γ defined in (10). Hence, to study the behavior of u near the origin, we may deal with the same question for v . More precisely, we want to prove that the weighted operator $L_\gamma v$ satisfies a suitable weak Harnack inequality. Notice that the natural functional framework for the new equation of v is the space $Y^{s,\gamma}(\mathbb{R}^N)$ defined in Section 2.

The statement of the result is the following.

Theorem 3.3. *(Weak Harnack inequality)*

Let $r > 0$ such that $B_{2r} \subset \Omega$. Assume that $f \geq 0$ and let $v \in Y^{s,\gamma}(\mathbb{R}^N)$ be a supersolution to (21) with $v \geq 0$ in \mathbb{R}^N . Then, for every $q < \frac{N}{N-2s}$ there exists a positive constant $C = C(N, s, \gamma)$ such that

$$\left(\int_{B_r} v^q d\mu \right)^{\frac{1}{q}} \leq C \inf_{B_{\frac{3}{2}r}} v.$$

The proof follows classical arguments by Moser and Krylov-Safonov (see [19] for the local case with weights). For the nonlocal case we have the precedent of [16], where the kernel is comparable to a fractional Laplacian and the operator considered is of fractional p -Laplacian type. Since the kernel defined in (10) is singular we have to check the arguments step by step. That is, our result can be seen as the fractional counterpart of [19]. Notice that it is enough to consider the case

$B_r(x_0) = B_r(0)$. For simplicity of typing, we will write B_r instead of $B_r(0)$. We start proving the following estimate.

Lemma 3.4. *Let $R > 0$ such that $B_R \subset \Omega$, and assume that $v \in Y^{s,\gamma}(\mathbb{R}^N)$ with $v \geq 0$, is a supersolution to (21). Let $k > 0$ and suppose that for some $\sigma \in (0, 1]$ we have*

$$(22) \quad |B_r \cap \{v \geq k\}|_{d\mu} \geq \sigma |B_r|_{d\mu}$$

with $0 < r < \frac{R}{16}$. Then there exists a positive constant $C = C(N, s, \gamma)$ such that

$$|B_{6r} \cap \{v \leq 2\delta k\}|_{d\mu} \leq \frac{C}{\sigma \log(\frac{1}{2\delta})} |B_{6r}|_{d\mu}$$

for all $\delta \in (0, \frac{1}{4})$.

Proof. Without loss of generality we can assume that $v > 0$ in B_R , (otherwise we can deal with $v + \varepsilon$ and let $\varepsilon \rightarrow 0$ at the end). Let $\psi \in C_0^\infty(B_R)$ be such that $0 \leq \psi \leq 1$, $\text{supp } \psi \subset B_{7r}$, $\psi = 1$ in B_{6r} and $|\nabla \psi| \leq \frac{C}{r}$.

Using $\psi^2 v^{-1}$ as a test function in (21), it follows that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (v(x) - v(y))(\psi^2(x)v^{-1}(x) - \psi^2(y)v^{-1}(y))d\nu \geq 0.$$

Thus

$$(23) \quad 0 \leq \int_{B_{8r}} \int_{B_{8r}} (v(x) - v(y)) \left(\frac{\psi^2(x)}{v(x)} - \frac{\psi^2(y)}{v(y)} \right) d\nu + 2 \int_{\mathbb{R}^N \setminus B_{8r}} \int_{B_{8r}} (v(x) - v(y)) \frac{\psi^2(x)}{v(x)} d\nu.$$

Denote $x = |x|x'$ and $y = \rho y'$, where $|x'| = |y'| = 1$ and $\rho := |y|$. We have that

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_{8r}} \int_{B_{8r}} (v(x) - v(y)) \frac{\psi^2(x)}{v(x)} d\nu &\leq \int_{\mathbb{R}^N \setminus B_{8r}} \int_{B_{8r}} \psi^2(x) d\nu \\ &\leq \int_{B_{7r}} \frac{\psi^2(x)}{|x|^\gamma} \int_{8r}^\infty \frac{\rho^{N-\gamma-1}}{|x|^{N+2s}} \left(\int_{\mathbb{S}^{N-1}} \frac{dy'}{|\frac{\rho}{|x|}y' - x'|^{N+2s}} \right) d\rho dx. \end{aligned}$$

Setting here $\tau := \frac{\rho}{|x|}$,

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_{8r}} \int_{B_{8r}} (v(x) - v(y)) \frac{\psi^2(x)}{v(x)} d\nu &\leq C \int_{B_{7r}} \frac{\psi^2(x)}{|x|^{2\gamma+2s}} \int_{\frac{8}{7}}^\infty \tau^{N-\gamma-1} \left(\int_{\mathbb{S}^{N-1}} \frac{dy'}{|\tau y' - x'|^{N+2s}} \right) d\tau dx \\ &\leq C \int_{B_{7r}} \frac{\psi^2(x)}{|x|^{2\gamma+2s}} \int_{\frac{8}{7}}^\infty \tau^{N-\gamma-1} D(\tau) d\tau dx \end{aligned}$$

where

$$D(\tau) := 2 \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \int_0^\pi \frac{\sin^{N-2}(\theta)}{(1 - 2\sigma \cos(\theta) + \tau^2)^{\frac{N+2s}{2}}} d\theta.$$

Considering the behavior of D near from 0, 1 and at ∞ (see [22]), we obtain that

$$\int_{\frac{8}{7}}^\infty \tau^{N-\gamma-1} D(\tau) d\tau \leq C,$$

and therefore we conclude that

$$(24) \quad \int_{\mathbb{R}^N \setminus B_{8r}} \int_{B_{8r}} (v(x) - v(y)) \frac{\psi^2(x)}{v(x)} d\nu \leq C r^{N-2s-2\gamma}.$$

Moreover

$$\begin{aligned}
(25) \quad & \int_{B_{8r}} \int_{B_{8r}} (v(x) - v(y)) \left(\frac{\psi^2(x)}{v(x)} - \frac{\psi^2(y)}{v(y)} \right) d\nu \\
&= \int_{B_{6r}} \int_{B_{6r}} (v(x) - v(y)) \left(\frac{\psi^2(x)}{v(x)} - \frac{\psi^2(y)}{v(y)} \right) d\nu \\
&\quad + \iint_{B_{8r} \times B_{8r} \setminus B_{6r} \times B_{6r}} (v(x) - v(y)) \left(\frac{\psi^2(x)}{v(x)} - \frac{\psi^2(y)}{v(y)} \right) d\nu \\
&\leq \int_{B_{6r}} \int_{B_{6r}} (v(x) - v(y)) \left(\frac{1}{v(x)} - \frac{1}{v(y)} \right) d\nu + Cr^{N-2s-2\gamma},
\end{aligned}$$

where the last inequality follows as a consequence of that $\psi \equiv 1$ in B_{6r} and that the integral in $B_{8r} \times B_{8r} \setminus B_{6r} \times B_{6r}$ can be estimated in the same way as (24).

Furthermore, from [17, Proof of Lemma 1.3], there exist $C_1, C_2 > 0$ such that

$$(26) \quad (v(x) - v(y)) \left(\frac{\psi^2(x)}{v(x)} - \frac{\psi^2(y)}{v(y)} \right) \leq -C_1(\log(v(x)) - \log(v(y)))^2 \psi^2(y) + C_2(\psi(x) - \psi(y))^2.$$

Hence from (26) we deduce

$$(27) \quad \int_{B_{6r}} \int_{B_{6r}} (v(x) - v(y)) \left(\frac{1}{v(x)} - \frac{1}{v(y)} \right) d\nu \leq -C_1 \int_{B_{6r}} \int_{B_{6r}} (\log(v(x)) - \log(v(y)))^2 d\nu,$$

and thus, putting together (23), (24), (25) and (27), it follows that

$$(28) \quad \int_{B_{6r}} \int_{B_{6r}} (\log(v(x)) - \log(v(y)))^2 d\nu \leq Cr^{N-2s-2\gamma}.$$

Let $\delta \in (0, 1/4)$. We set $w(x) := \min\{\log(\frac{1}{2\delta}), \log(\frac{k}{v})\}_+$, and hence, since w is a truncation of $\log(\frac{k}{v})$, from (28) we obtain that

$$\int_{B_{6r}} \int_{B_{6r}} (w(x) - w(y))^2 d\nu \leq Cr^{N-2s-2\gamma}.$$

Call

$$\langle w \rangle_{B_{6r}} := \frac{1}{|B_{6r}|_{d\mu}} \int_{B_{6r}} w(x) d\mu.$$

Thus, using Hölder and Poincaré-Wirtinger inequalities, Theorem 2.13,

$$(29) \quad \int_{B_{6r}} |w(x) - \langle w \rangle_{B_{6r}}| d\mu \leq C|B_{6r}|_{d\mu}.$$

Notice that $\{x \in \Omega : w(x) = 0\} = \{x \in \Omega : v(x) \geq k\}$, and then from (22) we have

$$|B_{6r} \cap \{w = 0\}|_{d\mu} \geq \frac{\sigma}{6^{N-2\gamma}} |B_{6r}|_{d\mu}.$$

As a consequence of this, it can be seen that

$$|B_{6r} \cap \{w = \log(\frac{1}{2\delta})\}|_{d\mu} \leq \frac{6^{N-2\gamma}}{\sigma \log(\frac{1}{2\delta})} \int_{B_{6r}} |w(x) - \langle w \rangle_{B_{6r}}| d\mu,$$

and hence, we conclude the result by applying (29) and the fact that

$$\{B_{6r} \cap \{v \leq 2\delta k\}\} = \{B_{6r} \cap \{w = \log(\frac{1}{2\delta})\}\}.$$

□

As a consequence we have the next estimate on $\inf_{B_{4r}} v$.

Lemma 3.5. *Assume that the hypotheses of Lemma 3.4 are satisfied. Then, there exists $\delta \in (0, \frac{1}{2})$, depending only on N, s, σ and γ , such that*

$$(30) \quad \inf_{B_{4r}} v \geq \delta k.$$

Proof. We set $w := (l - v)_+$ where $l \in (\delta k, 2\delta k)$ and let $\psi \in \mathcal{C}_0^\infty(B_\rho)$ with $r \leq \rho < 6r$.

Using $w\psi^2$ as a test function in (21) and using similar arguments to those in [16, Lemma 3.2], we reach that

$$(31) \quad \begin{aligned} & \int_{B_\rho} \int_{B_\rho} (w(x)\psi(x) - w(y)\psi(y))^2 d\nu \\ & \leq C_1 \int_{B_\rho} \int_{B_\rho} \max\{w(x), w(y)\}^2 (\psi(x) - \psi(y))^2 d\nu \\ & \quad + l^2 |B_\rho \cap \{v < l\}|_{d\mu} \times \sup_{\{x \in \text{supp}(\psi)\}} \int_{\mathbb{R}^N \setminus B_\rho} \frac{dy}{|x - y|^{N+2s}}. \end{aligned}$$

We define now the sequences $\{l_j\}_{j \in \mathbb{N}}$, $\{\rho_j\}_{j \in \mathbb{N}}$ and $\{\bar{\rho}_j\}_{j \in \mathbb{N}}$ by setting

$$l_j := \delta k + 2^{-j-1} \delta k, \quad \rho_j := 4r + 2^{1-j} r, \quad \bar{\rho}_j := \frac{\rho_j + \rho_{j+1}}{2}.$$

Likewise, let us denote

$$w_j := (l_j - v)_+, \quad B_j := B_{\rho_j},$$

and let $\psi_j \in \mathcal{C}_0^\infty(B_{\bar{\rho}_j})$ be such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ in B_{j+1} and $|\nabla \psi_j| \leq 2^{j+3}/r$.

Using the Sobolev inequality stated in Proposition 2.12 we obtain that

$$C(N, s, \gamma) \left(\int_{B_j} \frac{|w_j \psi_j|^{2_s^*}}{|x|^{\gamma 2_s^*}} dx \right)^{\frac{2}{2_s^*}} \leq \int_{B_j} \int_{B_j} (w_j(x)\psi_j(x) - w_j(y)\psi_j(y))^2 d\nu.$$

Hence, using the facts that

$$w_j \psi_j \geq (l_j - l_{j+1}) \text{ in } B_{j+1} \cap \{v < l_{j+1}\},$$

and

$$|x|^{-2_s^* \gamma} \geq \bar{C} r^{-(2_s^*-2)\gamma} |x|^{-2\gamma} \text{ in } B_j,$$

with \bar{C} independent of j , it follows that

$$\left(\int_{B_j} \frac{|w_j \psi_j|^{2_s^*}}{|x|^{\gamma 2_s^*}} dx \right)^{\frac{2}{2_s^*}} \geq \frac{C}{r^{(2_s^*-2)\gamma}} (l_j - l_{j+1})^2 |B_{j+1} \cap \{v < l_{j+1}\}|_{d\mu}^{\frac{2}{2_s^*}}.$$

Since $|B_{j+1}|_{d\mu} = C r^{N-2\gamma}$, then

$$\frac{r^{(2_s^*-2)\gamma}}{(|B_{j+1}|_{d\mu})^{\frac{2}{2_s^*}}} = C r^{-(N-2s-2\gamma)} r^{4s\gamma(\frac{1}{N-2s} - \frac{1}{N})} \leq C(N, s, \gamma, \Omega) r^{-(N-2s-2\gamma)}.$$

Hence we conclude that

$$\begin{aligned} & (l_j - l_{j+1})^2 \left(\frac{|B_{j+1} \cap \{v < l_{j+1}\}|_{d\mu}}{|B_{j+1}|_{d\mu}} \right)^{\frac{2}{2_s^*}} \\ & \leq C(N, s, \gamma, \Omega) r^{-(N-2s-2\gamma)} \int_{B_j} \int_{B_j} (w_j(x)\psi_j(x) - w_j(y)\psi_j(y))^2 d\nu. \end{aligned}$$

Applying (31) to w_j , we conclude that

$$\begin{aligned}
 (32) \quad & (l_j - l_{j+1})^2 \left(\frac{|B_{j+1} \cap \{v < j+1\}|_{d\mu}}{|B_{j+1}|_{d\mu}} \right)^{\frac{2}{2s}} \\
 & \leq \frac{C(N, s, \gamma)}{r^{(N-2s-2\gamma)}} \left(C_1 \int_{B_j} \int_{B_j} \max\{w_j(x), w_j(y)\}^2 (\psi_j(x) - \psi_j(y))^2 d\nu \right. \\
 & \quad \left. + l_j^2 |B_j \cap \{v < l_j\}|_{d\mu} \sup_{\{x \in \text{supp}(\psi_j)\}} \int_{\mathbb{R}^N \setminus B_j} \frac{dy}{|x-y|^{N+2s}} \right).
 \end{aligned}$$

We have

$$\begin{aligned}
 (33) \quad & \int_{B_j} \int_{B_j} \max\{w_j(x), w_j(y)\}^2 (\psi_j(x) - \psi_j(y))^2 d\nu \\
 & \leq l_j^2 \|\nabla \psi_j\|_{L^\infty(B_j)}^2 \int_{B_j \cap \{v < l_j\}} \frac{dx}{|x|^\gamma} \int_{B_j} \frac{|x-y|^{2-2s}}{|x-y|^N} \frac{dy}{|y|^\gamma} \\
 & \leq C 2^{2j} l_j^2 r^{-2s} \int_{B_j \cap \{v < l_j\}} \frac{dx}{|x|^{2\gamma}} = C 2^{2j} l_j^2 r^{-2s} |B_j \cap \{v < l_j\}|_{d\mu}.
 \end{aligned}$$

Now, estimating the term

$$\sup_{\{x \in \text{supp}(\psi_j)\}} \int_{\mathbb{R}^N \setminus B_j} \frac{dy}{|x-y|^{N+2s}}$$

as in [16, Lemma 3.2], and considering (32) and (33), we obtain that

$$\begin{aligned}
 (l_j - l_{j+1})^2 \left(\frac{|B_{j+1} \cap \{v < j+1\}|_{d\mu}}{|B_{j+1}|_{d\mu}} \right)^{\frac{2}{2s}} & \leq 2^{j(2+2s+N)} l_j^2 \frac{C(N, s, \gamma)}{r^{(N-2s-2\gamma)}} r^{-2s} |B_j \cap \{v < l_j\}|_{d\mu} \\
 & \leq \tilde{C} 2^{j(2+2s+N)} l_j^2 \frac{|B_j \cap \{v < j\}|_{d\mu}}{|B_j|_{d\mu}}
 \end{aligned}$$

where $\tilde{C} = \tilde{C}(N, s, \gamma)$ but independent of j and r .

Defining $A_j := \frac{|B_j \cap \{v < j\}|_{d\mu}}{|B_j|_{d\mu}}$ and following as in [16], we get the desired result. \square

Now, we need to obtain a kind of *reverse Hölder inequality* for v .

Lemma 3.6. *Let $r > 0$ such that $B_{3r/2} \subset \Omega$ and suppose that v is a supersolution to (21). Then, for every $0 < \alpha_1 < \alpha_2 < \frac{N}{N-2s}$, we have*

$$(34) \quad \left(\frac{1}{|B_r|_{d\mu}} \int_{B_r} v^{\alpha_2} d\mu \right)^{\frac{1}{\alpha_2}} \leq C \left(\frac{1}{|B_{3r/2}|_{d\mu}} \int_{B_{3r/2}} v^{\alpha_1} d\mu \right)^{\frac{1}{\alpha_1}},$$

with $C = C(N, s, \gamma, \alpha_1, \alpha_2) > 0$.

Proof. Let $q \in (1, 2)$ and $d > 0$. Set $\tilde{v} := (v + d)$, and assume that $\psi \in \mathcal{C}_0^\infty(\Omega)$ is such that $\text{supp}(\psi) \subset B_{\tau r}$, $\psi = 1$ in $B_{\tau' r}$ and $|\nabla \psi| \leq \frac{C}{(\tau - \tau')r}$ where $\frac{1}{2} \leq \tau' < \tau < \frac{3}{2}$. Then using $\tilde{v}^{1-q} \psi^2$ as a test function in (11), we obtain that

$$0 \leq \int_{B_{\tau r}} \int_{B_{\tau r}} (\tilde{v}(x) - \tilde{v}(y)) \left(\frac{\psi^2(x)}{\tilde{v}^{q-1}(x)} - \frac{\psi^2(y)}{\tilde{v}^{q-1}(y)} \right) d\nu + 2 \int_{\mathbb{R}^N \setminus B_{\tau r}} \int_{B_{\tau r}} (\tilde{v}(x) - \tilde{v}(y)) \frac{\psi^2(x)}{\tilde{v}^{q-1}(x)} d\nu.$$

Since $|x| < |y|$ in $B_{\tau r} \times (\mathbb{R}^N \setminus B_{\tau r})$, and using the positivity of \tilde{v} it follows

$$\int_{\mathbb{R}^N \setminus B_{\tau r}} \int_{B_{\tau r}} (\tilde{v}(x) - \tilde{v}(y)) \frac{\psi^2(x)}{\tilde{v}^{q-1}(x)} d\nu \leq \left(\int_{B_{\tau r}} \tilde{v}^{2-q} \psi^2 d\mu \right) \left(\sup_{\{x \in \text{supp}(\psi)\}} \int_{\mathbb{R}^N \setminus B_{\tau r}} \frac{dy}{|x-y|^{N+2s}} \right).$$

Furthermore, by the pointwise inequality of Lemma 3.3-(i) in [21], there exist positive constants C_1 and C_2 , depending on q , such that

$$\begin{aligned} \int_{B_{\tau r}} \int_{B_{\tau r}} (\tilde{v}(x) - \tilde{v}(y)) \left(\frac{\psi^2(x)}{\tilde{v}^{q-1}(x)} - \frac{\psi^2(y)}{\tilde{v}^{q-1}(y)} \right) d\nu \\ \leq -C_1 \int_{B_{\tau r}} \int_{B_{\tau r}} (\tilde{v}^{\frac{2-q}{2}}(x) \psi(x) - \tilde{v}^{\frac{2-q}{2}}(y) \psi(y))^2 d\nu \\ + C_2 \int_{B_{\tau r}} \int_{B_{\tau r}} ((\tilde{v}^{2-q}(x) + \tilde{v}^{2-q}(y))(\psi(x) - \psi(y))^2) d\nu. \end{aligned}$$

By symmetry we have

$$\int_{B_{\tau r}} \int_{B_{\tau r}} ((\tilde{v}^{2-q}(x) + \tilde{v}^{2-q}(y))(\psi(x) - \psi(y))^2) d\nu = 2 \int_{B_{\tau r}} \int_{B_{\tau r}} (\tilde{v}^{2-q}(x)(\psi(x) - \psi(y))^2) d\nu$$

and proceeding as in [4, Lemma 4.6] we obtain

$$\int_{B_{\tau r}} \int_{B_{\tau r}} ((\tilde{v}^{2-q}(x) + \tilde{v}^{2-q}(y))(\psi(x) - \psi(y))^2) d\nu \leq \frac{Cr^{-2s}}{(\tau - \tau')^2} \int_{B_{\tau r}} \tilde{v}^{2-q} d\mu.$$

Since

$$\sup_{\{x \in \text{Supp}(\psi)\}} \int_{\mathbb{R}^N \setminus B_{\tau r}} \frac{dy}{|x-y|^{N+2s}} \leq Cr^{-2s},$$

then combining the estimates above we reach that

$$\int_{B_{\tau r}} \int_{B_{\tau r}} (\tilde{v}^{\frac{2-q}{2}}(x) \psi(x) - \tilde{v}^{\frac{2-q}{2}}(y) \psi(y))^2 d\nu \leq \frac{Cr^{-2s}}{(\tau - \tau')^2} \int_{B_{\tau r}} \tilde{v}^{2-q} d\mu.$$

Hence, from the previous inequality and the Sobolev inequality in Proposition 2.12, we get

$$\begin{aligned} \left(\frac{1}{|B_{\tau' r}| d\mu} \int_{B_{\tau' r}} \tilde{v}^{\frac{(2-q)N}{N-2s}} d\mu \right)^{\frac{N-2s}{N}} &\leq \left(\frac{1}{|B_{\tau' r}| d\mu} \int_{B_{\tau r}} (\tilde{v}^{\frac{2-q}{2}} \psi)^{2^*} d\mu \right)^{\frac{N-2s}{N}} \\ &\leq \frac{C}{|B_{\tau r}| d\mu (\tau - \tau')^2} \int_{B_{\tau r}} \tilde{v}^{2-q} d\mu. \end{aligned}$$

Since $q \in (1, 2)$ is arbitrary and $\frac{N}{N-2s} > 1$ by using Hölder inequality we obtain the estimate (34) for $\tilde{v} = v + d$ with α_1 and α_2 in the hypotheses. Finally letting $d \rightarrow 0$ and by the *Monotone Convergence Theorem* we conclude. \square

In order to obtain the weak Harnack inequality, we need to prove the following estimate.

Lemma 3.7. *Let $r > 0$ such that $B_r \subset \Omega$. Assume that v is a supersolution to (21). Then, there exists a constant $\eta \in (0, 1)$ depending only on N , s and γ such that*

$$\left(\frac{1}{|B_r| d\mu} \int_{B_r} v^\eta d\mu \right)^{\frac{1}{\eta}} \leq C \inf_{B_r} v.$$

To prove Lemma 3.7 (see [26] and [16, Lemma 4.1]) we need the next covering result in the spirit of Krylov-Safonov theory. Notice that we are working with a doubling measure on bounded domains of \mathbb{R}^N .

Lemma 3.8. *Assume that $E \subset B_r(x_0)$ is a measurable set. For $\bar{\delta} \in (0, 1)$, we define*

$$[E]_{\bar{\delta}} := \bigcup_{\rho > 0} \{B_{3\rho}(x) \cap B_r(x_0), x \in B_r(x_0) : |E \cap B_{3\rho}(x)|_{d\mu} > \bar{\delta}|B_\rho(x)|_{d\mu}\}.$$

Then, either

- (1) $|[E]_{\bar{\delta}}|_{d\mu} \geq \frac{\tilde{C}}{\bar{\delta}}|E|_{d\mu}$, or
- (2) $[E]_{\bar{\delta}} = B_r(x_0)$,

where \tilde{C} depends only on N , s and γ .

Proof of Lemma 3.7. Notice that, for any $\eta > 0$,

$$(35) \quad \frac{1}{|B_r|_{d\mu}} \int_{B_r} v^\eta d\mu = \eta \int_0^\infty t^{\eta-1} \frac{|B_r \cap \{v > t\}|_{d\mu}}{|B_r|_{d\mu}} dt.$$

Then, for $t > 0$ and $i \in \mathbb{N}$, we set $A_t^i := \{x \in B_r : v(x) > t\delta^i\}$ where δ is given by Lemma 3.5. Notice that $A_t^{i-1} \subset A_t^i$.

Let $\rho > 0$ and $x \in B_r$ such that $B_{3\rho}(x) \cap B_r \subset [A_t^{i-1}]_{\bar{\delta}}$. Thus,

$$|A_t^{i-1} \cap B_{3\rho}(x)|_{d\mu} > \bar{\delta}|B_\rho|_{d\mu} = \frac{\bar{\delta}}{3^{N-2\gamma}}|B_{3\rho}|_{d\mu}.$$

Hence, using Lemma 3.5, we reach that

$$v(x) > \delta(t\delta^{i-1}) = t\delta^i \text{ for all } x \in B_r,$$

and therefore $[A_t^{i-1}]_{\bar{\delta}} \subset A_t^i$, being $[A_t^{i-1}]_{\bar{\delta}}$ as in Lemma 3.8. This fact, together with Lemma 3.8, allows us to deduce that

$$(36) \quad A_t^i = B_r \text{ or } |A_t^i|_{d\mu} \geq \frac{\tilde{C}}{\bar{\delta}}|A_t^{i-1}|_{d\mu}.$$

Thus, if for some $m \in \mathbb{N}$ we have

$$(37) \quad |A_t^0|_{d\mu} > \left(\frac{\bar{\delta}}{\tilde{C}}\right)^m |B_r|_{d\mu},$$

then $A_t^m = B_r$. If not, it follows from (36) that

$$|A_t^m|_{d\mu} \geq \frac{\tilde{C}}{\bar{\delta}}|A_t^{m-1}|_{d\mu}.$$

Since $A_t^{i-1} \subset A_t^m \subsetneq B_r$ for all $i \leq m$, the second point of the alternative (36) holds for A_t^{i-1} and then

$$|A_t^{m-1}|_{d\mu} \geq \frac{\tilde{C}}{\bar{\delta}}|A_t^{m-2}|_{d\mu} \dots \geq \left(\frac{\tilde{C}}{\bar{\delta}}\right)^{m-1} |A_t^0|_{d\mu} > \left(\frac{\tilde{C}}{\bar{\delta}}\right)^{-1} |B_r|_{d\mu}.$$

Thus $|A_t^m|_{d\mu} > |B_r|_{d\mu}$, a contradiction with the fact that $A_t^m \subsetneq B_r$. Hence $A_t^m = B_r$.

It is clear that (37) holds if

$$(38) \quad m > \frac{1}{\log(\frac{\tilde{C}}{\bar{\delta}})} \log \left(\frac{|A_t^0|_{d\mu}}{|B_r|_{d\mu}} \right),$$

and consequently, fixing m to be the smallest integer such that (38) holds, then $m \geq 1$ and

$$0 \leq m - 1 \leq \frac{1}{\log(\frac{\delta}{C})} \log \left(\frac{|A_t^0|_{d\mu}}{|B_r|_{d\mu}} \right).$$

Thus, using the fact that $\delta \in (0, \frac{1}{2})$, it can be checked that

$$\inf_{B_r} v > t\delta^m \geq t\delta \left(\frac{|A_t^0|_{d\mu}}{|B_r|_{d\mu}} \right)^{\frac{1}{\beta}},$$

with $\beta := \frac{\log(\frac{\delta}{C})}{\log(\delta)}$.

Set now $\xi := \inf_{B_r} v$. Then,

$$\frac{|B_r \cap \{v > t\}|_{d\mu}}{|B_r|_{d\mu}} = \frac{|A_t^0|_{d\mu}}{|B_r|_{d\mu}} \leq \tilde{C} \delta^{-\beta} t^{-\beta} \xi^\beta.$$

Going back to (35), we have

$$\begin{aligned} \frac{1}{|B_r|_{d\mu}} \int_{B_r} v^\eta d\mu &\leq \eta \int_0^a t^{\eta-1} dt + \eta \tilde{C} \int_a^\infty t^{\eta-1} \delta^{-\beta} t^{-\beta} \xi^\beta dt \\ &= a^\eta - \eta \tilde{C} \delta^{-\beta} \xi^\beta \frac{a^{\eta-\beta}}{\eta-\beta}. \end{aligned}$$

Choosing $a := \xi$ and $\eta := \frac{\beta}{2}$, we reach the result. \square

After this result, we can already prove the weighted weak Harnack inequality.

Proof of Theorem 3.3. Using Lemma 3.7 we obtain that

$$\left(\frac{1}{|B_r|_{d\mu}} \int_{B_r} v^\eta d\mu \right)^{\frac{1}{\eta}} \leq C \inf_{B_r} v$$

for some $\eta \in (0, 1)$. Fixing $1 \leq q < \frac{N}{N-2s}$, by Lemma 3.6 with $\alpha_1 = \eta$ and $\alpha_2 = q$, it follows that

$$\left(\frac{1}{|B_r|_{d\mu}} \int_{B_r} v^q d\mu \right)^{\frac{1}{q}} \leq C \left(\frac{1}{|B_{\frac{3}{2}r}|_{d\mu}} \int_{B_{\frac{3}{2}r}} v^\eta d\mu \right)^{\frac{1}{\eta}}.$$

Hence

$$\left(\frac{1}{|B_r|_{d\mu}} \int_{B_r} v^q d\mu \right)^{\frac{1}{q}} \leq C \inf_{B_{\frac{3}{2}r}} v$$

and we conclude. \square

As a consequence of the previous Harnack inequality, we get much information about the behavior of the supersolutions to (6) around the origin. In particular, we see that any of them must be unbounded, even if $f \in L^\infty(\Omega)$.

Lemma 3.9. *Let $\lambda \leq \Lambda_{N,s}$. Assume that u is a nonnegative function defined in Ω such that $u \not\equiv 0$, $u \in L^1(\Omega)$, $\frac{u}{|x|^{2s}} \in L^1(\Omega)$ and $u \geq 0$ in $\mathbb{R}^N \setminus \Omega$. If u satisfies $(-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} \geq 0$ in the weak sense in Ω , then there exists $\delta > 0$, and a constant $C = C(N, \delta, \gamma)$ such that for each ball $B_\delta(0) \subset\subset \Omega$,*

$$u \geq C|x|^{-\gamma} \text{ in } B_\delta(0),$$

where γ is defined in (19). In particular, for δ conveniently small, we can assume that $u > 1$ in $B_\delta(0)$.

Proof. Considering $v := |x|^\gamma u$, then $v \geq 0$ and it satisfies $L_\gamma v \geq 0$, with L_γ defined in (10). Hence using the weak Harnack inequality in Theorem 3.3, we conclude that $\inf_{B_r(0)} v \geq C$. Thus $u(x) \geq C|x|^{-\gamma}$ in $B_r(0)$ and the result follows. \square

4. OPTIMAL SUMMABILITY IN THE PRESENCE OF HARDY POTENTIAL

In this section we analyze the question of the optimal summability of the solution to the problem

$$(39) \quad \begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} &= f \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

with $0 < \lambda < \Lambda_{N,s}$.

4.1. Regularity of energy solution. Along this subsection we will assume that $f \in L^m(\Omega)$ with $m \geq \frac{2N}{N+2s}$, and thus the solution u will belong to $H_0^s(\Omega)$.

In particular, it is known that in the classical case, i.e. when $\lambda = 0$, if $m > \frac{N}{2s}$, then $u \in L^\infty(\Omega)$. However, as a consequence of Lemma 3.9, this feature is no longer true for $\lambda > 0$, and actually $u(x) \geq C|x|^{-\gamma}$ in a neighborhood of the origin. Hence, a natural question here is whether this rate is exactly the rate of growth of u , and the answer is yes for regular data, as the following theorem shows.

Theorem 4.1. *Let $f \in L^m(\Omega)$, $m > \frac{N}{2s}$. Let consider $u \in H_0^s(\Omega)$ the unique energy solution to problem (39), with $\lambda \leq \Lambda_{N,s}$, then $u(x) \leq C|x|^{-\gamma}$ in \mathbb{R}^N .*

Proof. Since the problem is linear, without loss of generality we can assume $f \geq 0$. Defining $v(x) := |x|^\gamma u(x)$, it can be checked that it solves

$$(40) \quad \begin{cases} L_\gamma(v) &= |x|^{-\gamma} f \text{ in } \Omega, \\ v &= 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where the operator $L_\gamma(v)$ was defined by (10). Consider now $G_k(v(x))$, specified in (14), with $k > 0$ as test function in (40). Hence,

$$\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{(v(x) - v(y))(G_k(v(x)) - G_k(v(y)))}{|x - y|^{N+2s}} \frac{dx}{|x|^\gamma} \frac{dy}{|y|^\gamma} = \int_\Omega |x|^{-\gamma} f G_k(v) dx.$$

Since for any $\sigma \in \mathbb{R}$, $\sigma = T_k(\sigma) + G_k(\sigma)$, then

$$\begin{aligned} & (v(x) - v(y))(G_k(v(x)) - G_k(v(y))) \\ &= (G_k(v(x)) - G_k(v(y)))^2 + (T_k(v(x)) - T_k(v(y)))(G_k(v(x)) - G_k(v(y))). \end{aligned}$$

Moreover, by [28, Lemma 2.5], we know that

$$(T_k(v(x)) - T_k(v(y)))(G_k(v(x)) - G_k(v(y))) \geq 0,$$

and therefore

$$\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|G_k(v(x)) - G_k(v(y))|^2}{|x - y|^{N+2s}} \frac{dx}{|x|^\gamma} \frac{dy}{|y|^\gamma} \leq \int_\Omega f G_k(v(x)) \frac{dx}{|x|^\gamma}.$$

Let us denote $A_k := \{x \in \Omega : v(x) \geq k\}$. Applying the weighted Sobolev inequality (Proposition 2.11) in the left hand side we obtain,

$$\mathcal{S} \|G_k(v)\|_{L^{2^*}(\Omega, |x|^{-\gamma} dx)}^2 \leq \int_{A_k} f G_k(v(x)) \frac{dx}{|x|^\gamma}$$

and using Hölder's inequality in the right hand side,

$$\left| \int_{A_k} f G_k(v(x)) \frac{dx}{|x|^\gamma} \right| \leq \|f\|_{L^m(\Omega)} \|G_k(v)\|_{L^{2_s^*}(\Omega, |x|^{-\gamma} dx)} |A_k|^{1-\frac{1}{2_s^*}-\frac{1}{m}}.$$

Thus we have that

$$\|G_k(v)\|_{L^{2_s^*}(\Omega, |x|^{-\gamma} dx)} \leq \mathcal{S}^{-1} \|f\|_{L^m(\Omega)} |A_k|^{1-\frac{1}{2_s^*}-\frac{1}{m}}.$$

On the other hand, since Ω is bounded, there exists a constant $c > 0$ such that $\|G_k(v)\|_{L^{2_s^*}(\Omega, |x|^{-\gamma} dx)} \geq c \|G_k(v)\|_{L^{2_s^*}(\Omega)}$. Moreover, for any $z > k$, we have that $A_z \subset A_k$ and $G_k(s) \chi_{A_z} \geq (z-k)$ for every $s \in \mathbb{R}$ and thus

$$(z-k) |A_z|^{\frac{1}{2_s^*}} \leq \frac{1}{c\mathcal{S}} \|f\|_{L^m(\Omega)} |A_k|^{1-\frac{1}{2_s^*}-\frac{1}{m}}.$$

Manipulating the above inequality we deduce that

$$|A_z| \leq \frac{\|f\|_{L^m(\Omega)}^{2_s^*} |A_k|^{2_s^*(1-\frac{1}{2_s^*}-\frac{1}{m})}}{(c\mathcal{S})^{2_s^*} (z-k)^{2_s^*}}.$$

Hence we apply Lemma 2.23 with the choice $\psi(s) := |A_s|$, using that

$$2_s^* \left(1 - \frac{1}{2_s^*} - \frac{1}{m}\right) > 1,$$

since $m > \frac{N}{2_s}$. Consequently there exists k_0 such that $\psi(k) \equiv 0$ for any $k \geq k_0$ and thus

$$\operatorname{ess\,sup}_\Omega v \leq k_0.$$

□

We next try to obtain under which conditions of λ the Calderón-Zygmund summability holds for the rest of the Lebesgue spaces contained in the dual of $H_0^s(\Omega)$, that is, $f \in L^m(\Omega)$ where $\frac{2N}{N+2s} \leq m < \frac{N}{2_s}$. Then we have the following result.

Theorem 4.2. *Let f be a positive function $f \in L^m(\Omega)$, with $\frac{2N}{N+2s} \leq m < \frac{N}{2_s}$. If*

$$(41) \quad \lambda < \Lambda_{N,s} \frac{4N(m-1)(N-2ms)}{m^2(N-2s)^2},$$

then there exists a constant $c = c(N, m, s) > 0$ such that the unique energy solution of problem (39) verifies

$$(42) \quad \|u\|_{L^{m_s^{**}}(\Omega)} \leq c \|f\|_{L^m(\Omega)} \quad \text{where} \quad m_s^{**} = \frac{mN}{N-2ms}.$$

Proof. Since $f \in H^{-s}(\Omega)$, the existence and uniqueness of an energy solution can be proved by means of a direct abstract Hilbert space approach.

To study the regularity of the solution, for every $k \in \mathbb{N}$, we consider $u_k \in L^\infty(\mathbb{R}^N) \cap H_0^s(\Omega)$, the solution to the following approximated problem

$$(43) \quad \begin{cases} (-\Delta)^s u_k - \lambda \frac{u_{k-1}}{|x|^{2s} + \frac{1}{k}} = f_k(x) & \text{in } \Omega, \\ u_k = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $f_k(x) := \min\{f(x), k\}$ and $u_0 = 0$.

In this way we obtain the following properties: (i) $\{u_k\}$ is an increasing sequence; (ii) each u_k is bounded, and (iii) $u_k \rightarrow u$, the unique solution to problem (6), in $L^p(\Omega)$, for every $1 \leq p \leq 2_s^*$.

Define $\beta = \frac{2_s^*}{2m' - 2_s^*} = \frac{N(m-1)}{N-2ms} \geq 1$, that satisfies $\beta m' = \frac{(\beta+1)}{2} 2_s^*$. Since u_k is bounded we can take u_k^β as a test function in (39), obtaining

$$\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{(u_k(x) - u_k(y))(u_k^\beta(x) - u_k^\beta(y))}{|x-y|^{N+2s}} dx dy \leq \lambda \int_\Omega \frac{u_k^{\beta+1}}{|x|^{2s}} dx + \int_\Omega f_k u_k^\beta dx.$$

By Hölder's inequality,

$$\int_\Omega f_k u_k^\beta dx \leq \|f_k\|_{L^m(\Omega)} \left(\int_\Omega u_k^{m'\beta} dx \right)^{\frac{1}{m'}} \leq \|f\|_{L^m(\Omega)} \left(\int_\Omega u_k^{\frac{(\beta+1)2_s^*}{2}} dx \right)^{\frac{\beta}{\beta+1} \frac{2}{2_s^*}}.$$

Now, by the algebraic inequality in (17), we get

$$(u_k(x) - u_k(y))(u_k^\beta(x) - u_k^\beta(y)) \geq \frac{4\beta}{(\beta+1)^2} (u_k^{\frac{\beta+1}{2}}(x) - u_k^{\frac{\beta+1}{2}}(y))^2,$$

and hence, using Hardy's inequality again, we conclude that

$$\frac{a_{N,s}}{2} \left(\frac{4\beta}{(\beta+1)^2} - \frac{\lambda}{\Lambda_{N,s}} \right) \iint_{D_\Omega} \frac{(u_k^{\frac{\beta+1}{2}}(x) - u_k^{\frac{\beta+1}{2}}(y))^2}{|x-y|^{N+2s}} dx dy \leq \|f\|_{L^m(\Omega)} \left(\int_\Omega u_k^{\frac{(\beta+1)2_s^*}{2}} dx \right)^{\frac{\beta}{\beta+1} \frac{2}{2_s^*}}.$$

On the other hand, hypothesis (41) is equivalent to

$$\left(\frac{4\beta}{(\beta+1)^2} - \frac{\lambda}{\Lambda_{N,s}} \right) > 0,$$

and thus, by the Sobolev inequality, we reach that

$$\left(\int_\Omega u_k^{\frac{(\beta+1)2_s^*}{2}} dx \right)^{\frac{2}{2_s^*(\beta+1)}} \leq C \|f\|_{L^m(\Omega)}.$$

Furthermore, $\frac{(\beta+1)2_s^*}{2} = \frac{mN}{N-2sm} = m_s^{**}$, and therefore passing to the limit we conclude. \square

Remark 4.3. Notice that, making $s \rightarrow 1$, the condition over λ becomes

$$\lambda < \frac{N(m-1)(N-2m)}{m^2},$$

the curve obtained in [14] for the local case.

4.2. About the optimality of the regularity results. For simplicity of typing we set

$$(44) \quad J_s(m) := \Lambda_{N,s} \frac{4N(m-1)(N-2ms)}{m^2(N-2s)^2}.$$

In [14], the authors proved that in the local case condition (41) (with $s = 1$) is optimal. In particular, they see that if $\lambda > J_1(m)$ and $\Omega = B_1(0)$, there exists a suitable radial function $f \in L^m(\Omega)$ such that the solution u does not belong to $L^{m_s^{**}}(\Omega)$.

In the nonlocal case the situation is more delicate. Indeed, we will see that in this case the previous example does not provide the optimality of the curve $J_s(m)$. It proves that the m_s^{**} summability does not hold above a curve, that we will call $P_s(m)$, that is in general above of $J_s(m)$. Thus, as far as we know, the optimality of $J_s(m)$ for every $\frac{2N}{N+2s} \leq m < \frac{N}{2s}$ remains open.

In order to define such curve $P_s(m)$, let us first recall that for all $\lambda \in (0, \Lambda_{N,s}]$ there exists a unique nonnegative constant $\alpha \in [0, \frac{N-2s}{2})$ such that

$$(45) \quad \lambda = \lambda(\alpha) = \frac{2^{2s} \Gamma(\frac{N+2s+2\alpha}{4}) \Gamma(\frac{N+2s-2\alpha}{4})}{\Gamma(\frac{N-2s+2\alpha}{4}) \Gamma(\frac{N-2s-2\alpha}{4})},$$

and $\lambda(\alpha)$ is a decreasing function of α (see [4]). Hence, for $m \in [\frac{2N}{N+2s}, \frac{N}{2s})$ fixed, we consider

$$(46) \quad \alpha_0(m) := \frac{N+2s}{2} - \frac{N}{m} \quad \text{and} \quad P_s(m) := \lambda(\alpha_0(m))$$

given by (45). Then we have the next result.

Lemma 4.4. *Assume that $f(x) = \frac{1}{|x|^\nu}$ with $\nu = \frac{N-\varepsilon}{m}$, for some $\varepsilon > 0$. Let $\lambda_1 \in (0, \Lambda_{N,s}]$ be such that $\lambda_1 \geq P_s(m)$. If u is the unique solution of (39) with $\lambda = \lambda_1$, then $u \notin L^{m^{**}}(B_1(0))$.*

Proof. Notice that $f \in L^m(B_1(0))$. From (45) we know that $\lambda_1 = \lambda(\alpha_1)$ for some $\alpha_1 \in [0, \frac{N-2s}{2})$. Since $\lambda_1 \geq P_s(m)$, using the fact that $\lambda(\alpha)$ is a decreasing function we reach that

$$\alpha_1 \leq \alpha_0(m) = \frac{N+2s}{2} - \frac{N}{m}.$$

Define

$$(47) \quad v(x) := C \left(\frac{1}{|x|^\gamma} - \frac{1}{|x|^{\nu-2s}} \right)$$

where $\gamma = \frac{N-2s}{2} - \alpha_1$. Since $\gamma \geq \frac{N}{m} - 2s$, using the fact that $\nu < \frac{N}{m}$ it follows that $\gamma > \nu - 2s$, and thus $v \geq 0$ in $B_1(0)$. Hence, choosing a suitable positive constant C , we reach that

$$(-\Delta)^s v - \lambda_1 \frac{v}{|x|^{2s}} = f \text{ in } B_1(0).$$

Since $v \leq 0$ in $\mathbb{R}^N \setminus B_1(0)$, by comparison it follows that $v \leq u$, where u is the unique solution of (39) for $\lambda = \lambda_1$. Since $\gamma \geq \frac{N}{m} - 2s$, then $v \notin L^{m^{**}}(B_1(0))$ and we conclude. \square

As a direct application of this lemma we can prove the optimality of the curve $J_s(m)$ in a particular case.

Lemma 4.5. *Assume that the hypotheses of Lemma 4.4 hold and let u be the solution of problem (39). If $m = \frac{2N}{N+2s}$ and $\lambda \geq J_s(m)$, then $u \notin L^{m^{**}}(B_1(0))$.*

Proof. Notice that $P_s(m)$, defined in (46), can be rewritten as

$$(48) \quad P_s(m) = \frac{2^{2s} \Gamma(\frac{N+2s}{2} - \frac{N}{2m}) \Gamma(\frac{N}{2m})}{\Gamma(\frac{N}{2} - \frac{N}{2m}) \Gamma(\frac{N}{2m} - s)},$$

and in the particular case of $m = \frac{2N}{N+2s}$, it satisfies

$$J_s(m) = P_s(m).$$

Hence $\lambda \geq P_s(m)$ by hypothesis, and we conclude applying Lemma 4.4. \square

Remark 4.6. Notice that in the local case, $s = 1$, $\alpha = \sqrt{\Lambda_{N,1} - \lambda}$ and $P_1(m) = J_1(m)$ for all $m \in [\frac{2N}{N+2}, \frac{N}{2})$, and thus Lemma 4.4 holds in the whole range.

Next we show that for radial functions the result in Lemma 4.4 cannot be improved. In other words the optimality of the curve $J_s(m)$ cannot be proved with radial functions. We start by proving the following result (for the properties of the Gamma function we refer to [9]).

Lemma 4.7. *Assume that $s \in (0, 1)$ and $m \in [\frac{2N}{N+2s}, \frac{N}{2s})$. Then*

$$(49) \quad J_s(m) \leq P_s(m)$$

and equality holds in (49) if and only if $m = \frac{2N}{N+2s}$.

Proof. Using (48) and the definition of $\Lambda_{N,s}$ (see (5)), it easily follows that (49) is equivalent to

$$(50) \quad D(m) := \frac{m^2}{(m-1)(N-2sm)} \frac{\Gamma(\frac{N+2s}{2} - \frac{N}{2m})\Gamma(\frac{N}{2m})}{\Gamma(\frac{N}{2} - \frac{N}{2m})\Gamma(\frac{N}{2m} - s)} \geq \frac{4N}{(N-2s)^2} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})} =: \Theta(N, s).$$

Furthermore, as we saw in the proof of Lemma 4.5,

$$D\left(\frac{2N}{N+2s}\right) = \Theta(N, s),$$

and thus (50) holds if we prove that D is an increasing function. Let us denote $D(m) = D_1(m)D_2(m)$ where

$$D_1(m) := \frac{\Gamma(\frac{N+2s}{2} - \frac{N}{2m})\Gamma(\frac{N}{2m})}{\Gamma(\frac{N}{2} - \frac{N}{2m})\Gamma(\frac{N}{2m} - s)} \quad \text{and} \quad D_2(m) := \frac{m^2}{(m-1)(N-2sm)}.$$

On the other hand, it is known that for $t > 0$ there holds $\Gamma'(t) = \psi(t)\Gamma(t)$ where $\psi(t)$, the so called Digamma function, is given by

$$\psi(t) := -\frac{1}{t} - C_0 + t \sum_{n=1}^{\infty} \frac{1}{n(n+t)},$$

with C_0 the Euler constant. Hence, it follows that

$$D'_1(m) = \frac{N}{2m^2} D_1(m) K(m),$$

where

$$K(m) = [\psi(a) - \psi(b) + \psi(c) - \psi(d)],$$

and

$$a := \frac{N+2s}{2} - \frac{N}{2m}, \quad b := \frac{N}{2} - \frac{N}{2m}, \quad c := \frac{N}{2m} - s, \quad d := \frac{N}{2m}.$$

Thus

$$D'(m) = D_1(m) \frac{m}{(m-1)(N-2sm)} \left(\frac{(m-2)N+2sm}{(m-1)(N-2sm)} + \frac{N}{2m} K(m) \right).$$

The first two terms here are positive, and then to analyze the sign of D' we have to study the function

$$H(m) := \left(\frac{(m-2)N+2sm}{(m-1)(N-2sm)} + \frac{N}{2m} K(m) \right).$$

By definition there holds

$$K(m) = -\frac{4sm^3}{N} \frac{(m-2)N+2sm}{(m-1)(N-2sm)((m-1)N+2sm)} + s \sum_{n=1}^{\infty} \left(\frac{1}{(n+a)(n+b)} - \frac{1}{(n+c)(n+d)} \right),$$

and noticing that $a = b + s$ and $d = c + s$, we have

$$\frac{1}{(n+a)(n+b)} = \frac{1}{s} \left(\frac{1}{n+b} - \frac{1}{n+b+s} \right)$$

and

$$\frac{1}{(n+c)(n+d)} = \frac{1}{s} \left(\frac{1}{n+c} - \frac{1}{n+c+s} \right).$$

Thus,

$$\begin{aligned} \frac{1}{(n+a)(n+b)} - \frac{1}{(n+c)(n+d)} &= \frac{1}{s} \left(\left(\frac{1}{n+b} - \frac{1}{n+c} \right) + \left(\frac{1}{n+c+s} - \frac{1}{n+b+s} \right) \right) \\ &= -\frac{b-c}{s} \left(\frac{1}{(n+c)(n+b)} - \frac{1}{(n+c+s)(n+b+s)} \right). \end{aligned}$$

Hence $H(m) = ((m-2)N + 2sm)H_1(m)$, where

$$H_1(m) := \frac{1}{(m-1)N + 2sm} - \frac{N}{4m^2} \sum_{n=1}^{\infty} \left[\frac{1}{(n+c)(n+b)} - \frac{1}{(n+c+s)(n+b+s)} \right].$$

Using the fact that $m \in (\frac{2N}{N+2s}, \frac{N}{2s})$, we obtain $(m-2)N + 2sm > 0$ and therefore the sign of H is the sign of H_1 . On the other hand, since $s \in (0, 1)$, then

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n+c)(n+b)} - \frac{1}{(n+c+s)(n+b+s)} \right] \leq \frac{1}{(1+c)(1+b)},$$

whence we conclude that

$$\begin{aligned} H_1(m) &\geq \frac{1}{(m-1)N + 2sm} - \frac{N}{4m^2} \frac{1}{(1+c)(1+b)} \\ &= \frac{1}{(m-1)N + 2sm} - \frac{N}{(N + 2m - 2sm)((m-1)N + 2m)}. \end{aligned}$$

Now, using the fact that $s \in (0, 1)$, we get $H_1(m) > 0$ and the result follows. \square

As a consequence we get the next regularity result.

Proposition 4.8. *Let $s \in (0, 1)$. Assume that $m \in (\frac{2N}{N+2s}, \frac{N}{2s})$, $s < 1$ and $f(x) = \frac{1}{|x|^\nu}$ with $\nu = \frac{N-\varepsilon}{m}$ for some small $\varepsilon > 0$. Then there exists $\tilde{\lambda} \in (J_s(m), P_s(m))$ such that the solution u of problem (39) with $\lambda = \tilde{\lambda}$ satisfies $u \in L^{m^{**}}(B_1(0))$.*

Proof. Fix $\delta > 0$ small enough so that if $\alpha \in (\alpha_0(m), \alpha_0(m) + \delta)$, then $\gamma := \frac{N-2s}{2} - \alpha$ satisfies $\gamma > \nu - 2s$ and $\gamma m_s^{**} < N$.

Define $\tilde{\lambda} = \lambda(\alpha)$. Since $\alpha > \alpha_0$ then $\tilde{\lambda} < P_s(m)$. Thus, due to the continuity of λ as a function of α , choosing δ small enough and applying Lemma 4.7 we deduce

$$J_s(m) < \tilde{\lambda} < P_s(m).$$

Let u be the unique solution of problem (39) with $\lambda = \tilde{\lambda}$ and consider the function v defined in (47). Since $\gamma > \nu - 2s$, by similar arguments as in the proof of Lemma 4.4 we conclude that $v \leq u$. By setting $w := u - v$, it follows that

$$\begin{cases} (-\Delta)^s w - \lambda \frac{w}{|x|^{2s}} = 0 & \text{in } \Omega, \\ w = -v \geq 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Thus, by Lemma 3.9 and Theorem 4.1, $w \simeq |x|^{-\gamma}$ close to the origin. Since also $v \simeq |x|^{-\gamma}$, also $u \simeq |x|^{-\gamma}$. By the definition of γ , we obtain that $\gamma m_s^{**} < N$, thus $u \in L^{m_s^{**}}(B_1(0))$ and then we conclude. \square

4.3. Nonvariational setting: weak solutions. In this subsection we consider $f \in L^m(\Omega)$, with $1 < m < \frac{2N}{N+2s}$.

Theorem 4.9. *Assume $1 < m < \frac{2N}{N+2s}$ and $\lambda < J_s(m)$ defined in (44). Then problem (39) has a unique weak solution u and it verifies*

$$(51) \quad \|u\|_{L^{m_s^{**}}(\Omega)} \leq c \|f\|_{L^m(\Omega)} \quad \text{where} \quad m_s^{**} = \frac{mN}{N-2ms}.$$

Moreover, $u \in W_0^{s_1, m_s^*}(\Omega)$, for all $s_1 < s$, with $m_s^* = \frac{mN}{N-ms}$.

Proof. Let $\{f_n\}_n \subset L^\infty(\Omega)$ be such that $0 \leq f_n \leq f$ and $f_n \uparrow f$ strongly in $L^m(\Omega)$. Define u_n to be the unique positive solution to the approximated problem

$$(52) \quad \begin{cases} (-\Delta)^s u_n - \lambda \frac{u_n}{|x|^{2s} + \frac{1}{n}} = f_n & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then $\{u_n\}_n$ is monotone in n . As in the proof of Theorem 4.2 we use u_n^β as a test function in (52) whit

$$0 < \beta := \frac{N(m-1)}{N-2ms} < 1$$

(actually we have to test with $(u_n + \delta)^\beta$, $\delta > 0$, and to make $\delta \rightarrow 0$ at the end, but to simplify we will drop this parameter here). Then,

$$(53) \quad \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{(u_n(x) - u_n(y))(u_n^\beta(x) - u_n^\beta(y))}{|x-y|^{N+2s}} dx dy \leq \lambda \int_\Omega \frac{u_n^{\beta+1}}{|x|^{2s}} dx + \int_\Omega f_n u_n^\beta dx.$$

By the fact that $m'\beta = \frac{(\beta+1)2_s^*}{2}$ and by Hölder's inequality,

$$(54) \quad \int_\Omega f_n u_n^\beta dx \leq \|f\|_{L^m(\Omega)} \left(\int_\Omega u_n^{\frac{(\beta+1)2_s^*}{2}} dx \right)^{\frac{\beta}{\beta+1} \frac{2}{2_s^*}}.$$

Now, using Lemma 2.22 and Hardy and Sobolev inequalities, it follows that

$$\frac{a_{N,s}}{2} \left(\frac{4\beta}{(\beta+1)^2} - \frac{\lambda}{\Lambda_{N,s}} \right) \left(\iint_{D_\Omega} \frac{(u_n^{\frac{\beta+1}{2}}(x) - u_n^{\frac{\beta+1}{2}}(y))^2}{|x-y|^{N+2s}} dx dy \right)^{\frac{1}{\beta+1}} \leq C(N, s) \|f\|_{L^m(\Omega)}.$$

Since $\lambda < J_s(m)$, noticing that $m_s^{**} = 2_s^* \frac{\beta+1}{2}$ and applying Sobolev inequality, (51) follows.

Furthermore, in particular

$$(55) \quad \int_\Omega u_n^{\frac{(\beta+1)2_s^*}{2}} dx \leq C_1 \quad \text{and} \quad \int_\Omega \frac{u_n^{\beta+1}}{|x|^{2s}} dx \leq C_2,$$

where C_1 and C_2 are independent of n . Since $\frac{(\beta+1)2_s^*}{2} > 1$, the Lebesgue Theorem implies that $u_n \uparrow u$ a.e. and strongly in $L^\sigma(\Omega)$ for all $1 \leq \sigma \leq \frac{(\beta+1)2_s^*}{2}$ and $\frac{u_n}{|x|^{2s}} \uparrow \frac{u}{|x|^{2s}}$ strongly in $L^1(\Omega)$.

Therefore, u is a weak solution of (39).

Moreover by using Fatou's Lemma, (54) and (55) in (53) we obtain

$$(56) \quad \iint_{D_\Omega} \frac{(u(x) - u(y))(u^\beta(x) - u^\beta(y))}{|x-y|^{N+2s}} dx dy \leq C.$$

Moreover u is the unique weak solution. Indeed if u_2 is an other solution of (39) with the above regularities, then setting $w = u_2 - u$, we conclude that

$$\begin{cases} (-\Delta)^s w - \lambda \frac{w}{|x|^{2s}} = 0 & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By testing with $\phi \in \mathcal{T}$ defined in (8), with $(-\Delta)^s \phi = \varphi > 0$, we obtain that $u_2 \equiv u$.

To finish we prove the regularity of the *fractional gradient*. Fix $s_1 < s$ and let $q = m_s^* < 2$. Call

$$d\sigma := \begin{cases} \left(\frac{u^\beta(x) - u^\beta(y)}{u(x) - u(y)} \right) dx dy & \text{if } u(x) \neq u(y), \\ 0 & \text{if } u(x) = u(y), \end{cases}$$

and notice that $d\sigma$ is positive. Therefore, by Hölder's inequality

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^q}{|x - y|^{N+qs_1}} dx dy &= \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^q}{|x - y|^{N+qs_1}} \left(\frac{u^\beta(x) - u^\beta(y)}{u(x) - u(y)} \right) \times \left(\frac{u(x) - u(y)}{u^\beta(x) - u^\beta(y)} \right) dx dy \\ &\leq \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} d\sigma \right)^{\frac{q}{2}} \times \left(\int_{\Omega} \int_{\Omega} \left(\frac{u(x) - u(y)}{u^\beta(x) - u^\beta(y)} \right)^{\frac{2}{2-q}} \frac{d\sigma}{|x - y|^{N-\theta}} \right)^{\frac{2-q}{2}} \\ &= \left(\int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(u^\beta(x) - u^\beta(y))}{|x - y|^{N+2s}} dx dy \right)^{\frac{q}{2}} \times \left(\int_{\Omega} \int_{\Omega} \left(\frac{u(x) - u(y)}{u^\beta(x) - u^\beta(y)} \right)^{\frac{q}{2-q}} \frac{dx dy}{|x - y|^{N-\theta}} \right)^{\frac{2-q}{2}}, \end{aligned}$$

where $\theta := \frac{2(s-s_1)}{2-q}$. The first term is bounded by (56), and the second one can be estimated as follows. Since $\beta < 1$, then

$$0 \leq \frac{u(x) - u(y)}{u^\beta(x) - u^\beta(y)} \leq \frac{1}{\beta} (u^{1-\beta}(x) + u^{1-\beta}(y)),$$

and hence

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \left(\frac{u(x) - u(y)}{u^\beta(x) - u^\beta(y)} \right)^{\frac{q}{2-q}} \frac{dx dy}{|x - y|^{N-\theta}} &\leq \frac{C}{\beta} \int_{\Omega} \int_{\Omega} ((u^{\frac{(1-\beta)q}{2-q}}(x) + u^{\frac{(1-\beta)q}{2-q}}(y)) \frac{dx dy}{|x - y|^{N-\theta}} \\ &\leq \frac{2C}{\beta} \int_{\Omega} u^{\frac{(1-\beta)q}{2-q}}(x) \left(\int_{\Omega} \frac{dy}{|x - y|^{N-\theta}} \right) dx. \end{aligned}$$

Notice that

$$\sup_{\{x \in \Omega\}} \left(\int_{\Omega} \frac{dy}{|x - y|^{N-\theta}} \right) \leq C,$$

hence

$$\int_{\Omega} \int_{\Omega} \left(\frac{u(x) - u(y)}{u^\beta(x) - u^\beta(y)} \right)^{\frac{q}{2-q}} \frac{dx dy}{|x - y|^{N-\theta}} \leq C_1 \int_{\Omega} u^{\frac{(1-\beta)q}{2-q}}(x) dx.$$

Now, since $\frac{(1-\beta)q}{2-q} = m_s^{**}$, the result follows using (51). \square

In the case where no condition is imposed on λ , then additional condition on f is needed. The next Theorem gives a necessary and sufficient condition to ensure the existence of a weak solution.

Theorem 4.10. *Let $\lambda \leq \Lambda_{N,s}$ and suppose that $f \in L^1(\Omega)$, $f \geq 0$. Then u is a positive weak solution to the problem (39) if and only if f satisfies*

$$\int_{B_r(0)} |x|^{-\gamma} f dx < +\infty,$$

for some $B_r(0) \subset\subset \Omega$.

Moreover, if u is the unique weak solution to (39), then

$$(57) \quad \forall k \geq 0 \quad T_k(u) \in H_0^s(\Omega), \quad u \in L^q(\Omega), \quad \forall q \in \left(1, \frac{N}{N-2s}\right),$$

$$(58) \quad |(-\Delta)^{\frac{s}{2}}(u)| \in L^r(\Omega), \quad \forall r \in \left(1, \frac{N}{N-s}\right),$$

and $u \in W_0^{s_1, q_1}(\Omega)$ for all $s_1 < s$ and for all $q_1 < \frac{N}{N-s}$.

Proof. Necessary condition: Consider u a positive weak solution to the problem (39) and consider $\varphi_n \in L^\infty(\Omega) \cap H_0^s(\Omega)$ the positive solution to

$$\begin{cases} (-\Delta)^s \varphi_n &= \lambda \frac{\varphi_{n-1}}{|x|^{2s} + \frac{1}{n}} + 1 \quad \text{in } \Omega, \\ \varphi_n &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

$$\begin{cases} (-\Delta)^s \varphi_0 &= 1 \quad \text{in } \Omega, \\ \varphi_0 &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then it is easy to check that $\varphi_0 \leq \varphi_1 \leq \varphi_{n-1} \leq \varphi_n \leq \varphi$, where φ is the pointwise limit and then

$$(59) \quad \begin{cases} (-\Delta)^s \varphi &= \lambda \frac{\varphi}{|x|^{2s}} + 1 \quad \text{in } \Omega, \\ \varphi &> 0 \quad \text{in } \Omega, \\ \varphi &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega). \end{cases}$$

Taking φ_n as a test function in (39), we get

$$\int_{\Omega} f \varphi_n dx \leq \int_{\Omega} u dx = C < \infty.$$

Hence, $\{f \varphi_n\}_{n \in \mathbb{N}}$ is an increasing sequence uniformly bounded in $L^1(\Omega)$, and then applying the Monotone Convergence Theorem and Lemma 3.9 we obtain

$$\tilde{C} \int_{B_r(0)} |x|^{-\gamma} f dx \leq \int_{\Omega} f \varphi dx \leq C.$$

Sufficient condition: Assume that

$$\int_{B_r(0)} |x|^{-\gamma} f dx < +\infty,$$

for all $B_r(0) \subset \subset \Omega$ small enough; let consider the sequence of energy solutions $u_n \in L^\infty(\Omega) \cap H_0^s(\Omega)$ to the following approximated problems

$$(60) \quad \begin{cases} (-\Delta)^s u_n &= \lambda \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} + f_n \quad \text{in } \Omega, \\ u_n(x) &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

$$\begin{cases} (-\Delta)^s u_0 &= f_1 \quad \text{in } \Omega, \\ u_0(x) &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega), \end{cases}$$

with $f_n = T_n(f)$ and $u_0 \leq u_1 \leq u_{n-1} \leq u_n$ in \mathbb{R}^N . Since $f_n \geq 0$, $u_n(x) \geq 0$ in Ω . Take $\varphi \in H_0^s(\Omega)$, the positive energy solution to (59), as a test function in (60). As a consequence of Lemma 4.1, it follows that

$$\int_{\Omega} u_n dx \leq \int_{\Omega} f \varphi dx \leq \tilde{C} \int_{\Omega} f |x|^{-\gamma} \leq C.$$

Hence, since the sequence $\{u_n\}_{n \in \mathbb{N}}$ is increasing, we can define $u := \lim_{n \rightarrow \infty} u_n$, and conclude that $u \in L^1(\Omega)$. We claim that $\frac{u}{|x|^{2s}} \in L^1(\Omega)$. Indeed, let ψ be the unique bounded positive solution to the problem

$$\begin{cases} (-\Delta)^s \psi &= 1 & \text{in } \Omega, \\ \psi &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

then $\psi \geq C$ in $B_r(0)$. By using ψ as a test function in (60),

$$\int_{\Omega} \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} dx \leq \frac{1}{C} \int_{B_r(0)} \frac{\psi u_{n-1}}{|x|^{2s} + \frac{1}{n}} dx + C(r) \int_{\Omega \setminus B_r(0)} u_n dx \leq C,$$

and thus

$$\lambda \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} + f_n \nearrow \lambda \frac{u}{|x|^{2s}} + f \text{ strongly in } L^1(\Omega).$$

Testing with $T_k(u_n)$ in (60) and considering the previous estimates, we easily get that $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $H_0^s(\Omega)$. Since the sequence $\{\lambda \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} + f_n\}_{n \in \mathbb{N}}$ converges strongly in $L^1(\Omega)$, then by the results of [28], we reach that $u \in L^\sigma(\Omega)$ for all $\sigma < \frac{N}{N-2s}$ and $|(-\Delta)^{\frac{s}{2}}(u)| \in L^r(\Omega)$ for all $r \in (1, \frac{N}{N-s})$. Moreover, according to Theorem 5 (C) of Chapter 5 in [33], we conclude that $u \in W_0^{s_1, q_1}(\Omega)$ for all $s_1 < s$ and for all $q_1 < \frac{N}{N-s}$. See too [27] and [2] for a simple proof. \square

Remark 4.11. As a consequence of this result, together with the weak Harnack inequality, one can easily prove nonexistence for $\lambda > \Lambda_{N,s}$.

5. PROBLEMS WITH THE HARDY POTENTIAL AND NONLINEAR TERM SINGULAR AT THE BOUNDARY.

The results in this section have some partial precedents in [1] and are also applicable to the local case. The aim will be to study the problem

$$(61) \quad \begin{cases} (-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + \frac{h(x)}{u^\sigma} & \text{in } \Omega, \\ u &> 0 & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where h is a nonnegative function, $\sigma > 0$ and $\lambda \geq 0$. As we pointed out in Remark 4.11 it can be easily checked that problem (61) has no positive solution for $\lambda > \Lambda_{N,s}$. Hence we will assume $\lambda \leq \Lambda_{N,s}$.

Remark 5.1. Call $\mu := -\sigma$. We know that problem (61) has no positive solution for $\mu > p_+(\lambda) := 1 + \frac{2s}{\gamma}$, where γ is defined in (20). A quite complete study is done in [11], also for the case $1 < \mu < p_+(\lambda)$ (see [20] for a different approach). The case $\mu = 1$ is related to the first eigenvalue of the operator $(-\Delta)^s(\cdot) - \lambda \frac{(\cdot)}{|x|^{2s}}$ and the case $0 \leq \mu < 1$ is easily handled as a minimization problem.

Therefore, finding a solution of (61) can be seen as proving that there is not a lower threshold for the power to solve the semilinear problem.

The main existence result in this section is the following.

Theorem 5.2. *Assume that $\sigma \geq 1$ and $\lambda \leq \Lambda_{N,s}$. Then, for all $h \in L^1(\Omega)$, problem (61) has a positive weak solution. More precisely,*

- (1) if $\sigma = 1$, then $u \in H_0^s(\Omega)$ whether $\lambda < \Lambda_{N,s}$, and $u \in W_0^{s,q}(\Omega)$ for all $q < 2$ if $\lambda = \Lambda_{N,s}$;
(2) if $\sigma > 1$, then $u \in H_{loc}^s(\Omega)$ with $G_k(u) \in H_0^s(\Omega)$ and $T_k^{\frac{\sigma+1}{2}}(u) \in H_0^s(\Omega)$. Moreover if $\left[\frac{4\sigma}{(\sigma+1)^2} - \frac{\lambda}{\Lambda_{N,s}} \right] > 0$, then $u^{\frac{\sigma+1}{2}} \in H_0^s(\Omega)$.

Proof. Let $h_n := T_n(h)$, the usual truncation of h , and define u_n to be the unique positive solution to the approximated problem

$$(62) \quad \begin{cases} (-\Delta)^s u_n &= \lambda \frac{u_n}{|x|^{2s}} + \frac{h_n(x)}{(u_n + \frac{1}{n})^\sigma} & \text{in } \Omega, \\ u_n &> 0 & \text{in } \Omega, \\ u_n &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega). \end{cases}$$

The existence follows by minimization and the uniqueness by using the result in Lemma 2.17. Since $T_n(h)$ is an increasing function in n , again by Lemma 2.17 we conclude that $\{u_n\}_{n \in \mathbb{N}}$ is an increasing function in n . We divide the proof in two cases.

First case: $\sigma = 1$ and $\lambda < \Lambda_{N,s}$.

Taking u_n as a test function in (62) and using the Hardy inequality, we obtain

$$\frac{a_{N,s}}{2} \left(1 - \frac{\lambda}{\Lambda_{N,s}} \right) \|u_n\|_{H_0^s(\Omega)}^2 \leq \int_{\Omega} \frac{h_n u_n}{u_n + \frac{1}{n}} dx \leq \int_{\Omega} h dx = C.$$

Thus $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $H_0^s(\Omega)$ and then there exists $u \in H_0^s(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H_0^s(\Omega)$ and $u_n \uparrow u$ strongly in $L^\eta(\Omega)$ for all $\eta < 2_s^*$.

Since $(-\Delta)^s u_n \geq 0$, using the monotonicity of $\{u_n\}_{n \in \mathbb{N}}$ and the compactness Lemma 2.18 we easily obtain that $u_n \rightarrow u$ strongly in $H_0^s(\Omega)$. Hence we conclude that u solves problem (61).

Second case: $\sigma > 1$.

Using now $G_k(u_n)$ as a test function in (62) we have

$$\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|G_k(u_n(x)) - G_k(u_n(y))|^2}{|x - y|^{N+2s}} dx dy - \lambda \int_{\Omega} \frac{u_n G_k(u_n)}{|x|^{2s}} dx \leq \int_{\Omega} \frac{h_n G_k(u_n)}{(u_n + \frac{1}{n})^\sigma} dx$$

and

$$\int_{\Omega} \frac{h_n G_k(u_n)}{(u_n + \frac{1}{n})^\sigma} dx \leq \frac{1}{k^{\sigma-1}} \int_{\Omega} h dx.$$

Moreover, $u_n G_k(u_n) = G_k^2(u_n) + k G_k(u_n)$, and thus

$$\begin{aligned} \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|G_k(u_n(x)) - G_k(u_n(y))|^2}{|x - y|^{N+2s}} dx dy - \lambda \int_{\Omega} \frac{G_k^2(u_n)}{|x|^{2s}} dx \\ \leq \lambda k \int_{\Omega} \frac{G_k(u_n)}{|x|^{2s}} dx + \frac{1}{k^{\sigma-1}} \int_{\Omega} h dx. \end{aligned}$$

Taking into account that $\lambda < \Lambda_{N,s}$, by the Hardy-Sobolev inequality we obtain

$$C \iint_{D_\Omega} \frac{|G_k(u_n(x)) - G_k(u_n(y))|^2}{|x - y|^{N+2s}} dx dy \leq \lambda k \int_{\Omega} \frac{G_k(u_n)}{|x|^{2s}} dx + C(k, h),$$

and applying Young and Hardy-Sobolev inequalities on the integral in the right hand side we reach that

$$\iint_{D_\Omega} \frac{|G_k(u_n(x)) - G_k(u_n(y))|^2}{|x - y|^{N+2s}} dx dy \leq C(k, \lambda, \Lambda_{N,s}, h).$$

Therefore $\{G_k(u_n)\}_{n \in \mathbb{N}}$ is uniformly bounded in $H_0^s(\Omega)$, and again by the Hardy-Sobolev inequality,

$$\int_{\Omega} \frac{G_k^2(u_n(x))}{|x|^{2s}} dx \leq C(k, \lambda, \Lambda_{N,s}, h).$$

Then we get

$$\int_{\Omega} \frac{u_n^2(x)}{|x|^{2s}} dx = \int_{\Omega} \frac{T_k^2(u_n(x))}{|x|^{2s}} dx + \int_{\Omega} \frac{G_k^2(u_n(x))}{|x|^{2s}} dx + 2 \int_{\Omega} \frac{T_k(u_n)G_k(u_n)}{|x|^{2s}} dx \leq C(k, \lambda, \Lambda_{N,s}, h).$$

Likewise, testing with $T_k^\sigma(u_n)$ in (62), it follows that

$$\begin{aligned} \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{(T_k^\sigma(u_n(x)) - T_k^\sigma(u_n(y)))(u_n(x) - u_n(y))}{|x - y|^{N+2s}} dx dy \\ \leq \lambda \int_{\Omega} \frac{u_n T_k^\sigma(u_n)}{|x|^{2s}} dx + \int_{\Omega} \frac{h_n T_k^\sigma(u_n)}{(u_n + \frac{1}{n})^\sigma} dx \\ \leq k^{\sigma-1} \lambda \int_{\Omega} \frac{u_n^2}{|x|^{2s}} dx + \int_{\Omega} h_n dx \leq C(k, \lambda, \Lambda_{N,s}, h), \end{aligned}$$

and applying Lemma 2.22 we conclude

$$\iint_{D_\Omega} \frac{(T_k^{\frac{\sigma+1}{2}}(u_n(x)) - T_k^{\frac{\sigma+1}{2}}(u_n(y)))^2}{|x - y|^{N+2s}} dx dy \leq C(k, \lambda, \Lambda_{N,s}, h, \sigma).$$

Thus $\{T_k^{\frac{\sigma+1}{2}}(u_n)\}_{n \in \mathbb{N}}$ is bounded in $H_0^s(\Omega)$. Furthermore, the strong maximum principle provides that

$$u_n \geq u_1 \geq c(K) > 0, \text{ for any compact set } K \subset \Omega.$$

Claim.- $\{T_k(u_n)\}_{n \in \mathbb{N}}$ is bounded in $H_{loc}^s(\Omega)$.

Since $\{u_n\}_{n \in \mathbb{N}}$ is an increasing sequence, then $T_k(u_n) \geq T_k(u_1)$, and for all $\Omega' \subset \subset \Omega$, $u_1 \geq C(\Omega')$. Thus,

$$T_k(u_n) \geq \min\{k, C(\Omega')\} =: C_0.$$

For $(x, y) \in \Omega' \times \Omega'$, we define $\alpha_n := \frac{T_k(u_n(x))}{C_0}$ and $\beta_n := \frac{T_k(u_n(y))}{C_0}$. It is clear that $\alpha_n, \beta_n \geq 1$. Therefore the following inequality holds,

$$(63) \quad (\alpha_n - \beta_n)^2 \leq (\alpha_n^{\frac{\sigma+1}{2}} - \beta_n^{\frac{\sigma+1}{2}})^2.$$

Indeed, if $\alpha_n = \beta_n$ the estimate is trivial. Otherwise, without loss of generality we can assume $\alpha_n > \beta_n \geq 1$. Let $0 < x := \frac{\beta_n}{\alpha_n} < 1$. Since $\sigma > 1$, we easily obtain that

$$0 \leq 1 - x \leq 1 - x^{\frac{\sigma+1}{2}},$$

and hence

$$(1 - x)^2 \leq (1 - x^{\frac{\sigma+1}{2}})^2.$$

Clearly $\alpha_n^2 < \alpha_n^{\sigma+1}$, and thus

$$\alpha_n^2 (1 - x)^2 \leq \alpha_n^{\sigma+1} (1 - x^{\frac{\sigma+1}{2}})^2.$$

Recalling the definition of x , (63) follows.

Finally, by the definition of α_n and β_n , we conclude that for $(x, y) \in \Omega' \times \Omega'$, we have

$$(T_k(u_n(x)) - T_k(u_n(y)))^2 \leq C_0^{1-\sigma} (T_k^{\frac{\sigma+1}{2}}(u_n(x)) - T_k^{\frac{\sigma+1}{2}}(u_n(y)))^2.$$

Thus the claim follows using the boundedness of $\{T_k^{\frac{\sigma+1}{2}}(u_n)\}_{n \in \mathbb{N}}$ in $H_0^s(\Omega)$.

Hence combining the estimates above, we obtain that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $H_{loc}^s(\Omega)$ and then, up to a subsequence, there exists $u \in H_{loc}^s(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \text{ weakly in } H_{loc}^s(\Omega), \\ G_k(u_n) &\rightharpoonup G_k(u) \text{ weakly in } H_0^s(\Omega) \text{ and} \\ T_k^{\frac{\sigma+1}{2}}(u) &\rightharpoonup T_k^{\frac{\sigma+1}{2}}(u) \text{ weakly in } H_0^s(\Omega). \end{aligned}$$

Applying the compactness result in Lemma 2.19 we obtain that $u_n \rightarrow u$ strongly in $H_{loc}^s(\Omega)$. Thus $u_n \uparrow u$ strongly in $L^r(\Omega)$ for all $1 \leq r < 2_s^*$.

Let $\phi \in \mathcal{T}$, where \mathcal{T} was defined in (8). Testing with ϕ in (62), it follows that

$$(64) \quad \int_{\Omega} (-\Delta)^s u_n \phi dx = \lambda \int_{\Omega} \frac{u_n \phi}{|x|^{2s}} dx + \int_{\Omega} \frac{\phi h_n(x)}{(u_n + \frac{1}{n})^\sigma} dx.$$

By the estimates above we reach that, when n goes to $+\infty$,

$$\lambda \int_{\Omega} \frac{u_n \phi}{|x|^{2s}} + \int_{\Omega} \frac{\phi h_n(x)}{(u_n + \frac{1}{n})^\sigma} dx \rightarrow \lambda \int_{\Omega} \frac{u \phi}{|x|^{2s}} + \int_{\Omega} \frac{\phi h(x)}{u^\sigma} dx < +\infty.$$

Moreover, we have

$$\int_{\Omega} (-\Delta)^s u_n \phi dx = \int_{\Omega} u_n (-\Delta)^s \phi dx \rightarrow \int_{\Omega} u (-\Delta)^s \phi dx,$$

as $n \rightarrow +\infty$. Therefore, passing to the limit in (64),

$$\int_{\Omega} (-\Delta)^s u \phi dx = \lambda \int_{\Omega} \frac{u \phi}{|x|^{2s}} dx + \int_{\Omega} \frac{h(x) \phi}{u^\sigma} dx,$$

i.e., u is a weak solution.

Finally, for every $\lambda < \Lambda_{N,s}$ take σ such that $\left[\frac{4\sigma}{(\sigma+1)^2} - \frac{\lambda}{\Lambda_{N,s}} \right] > 0$. By using u_n^σ as a test function in (62), it follows that

$$\iint_{D_\Omega} \frac{(u_n(x) - u_n(y))(u_n^\sigma(x) - u_n^\sigma(y))}{|x - y|^{N+2s}} dx dy \leq \lambda \int_{\Omega} \frac{u_n^{\sigma+1}}{|x|^{2s}} dx + \int_{\Omega} h_n dx.$$

By Lemma (2.22), we get

$$(u_n(x) - u_n(y))(u_n^\sigma(x) - u_n^\sigma(y)) \geq \frac{4\sigma}{(\sigma+1)^2} (u_n^{\frac{\sigma+1}{2}}(x) - u_n^{\frac{\sigma+1}{2}}(y))^2,$$

and hence, by the Hardy inequality,

$$\frac{a_{N,s}}{2} \left(\frac{4\sigma}{(\sigma+1)^2} - \frac{\lambda}{\Lambda_{N,s}} \right) \iint_{D_\Omega} \frac{(u_n^{\frac{\sigma+1}{2}}(x) - u_n^{\frac{\sigma+1}{2}}(y))^2}{|x - y|^{N+2s}} dx dy \leq \|h\|_{L^1(\Omega)}.$$

Therefore $u^{\frac{\sigma+1}{2}} \in H_0^s(\Omega)$ and this is the sense how the boundary value is reached. \square

We now deal with the case $\sigma < 1$. If $h \in L^{(\frac{2_s^*}{1-\sigma s})'}(\Omega)$, the existence of a positive energy solution can be proved proceeding as in the case $\sigma = 1$. However, our goal from now on will be to study the solvability when h has less regularity. Indeed, we have the following result.

Theorem 5.3. *Assume $\sigma < 1$, $\lambda < \Lambda_{N,s}$ and $h \in L^1(\Omega, |x|^{-(1-\sigma)\gamma} dx)$, $h \gneq 0$. Then problem (61) has at least a weak solution.*

Proof. Let $\{h_n\}_{n \in \mathbb{N}}$ be such that $h_n \geq 0$ and $h_n \uparrow h$ strongly in $L^1(\Omega)$. Define u_n as the unique positive solution to the approximated problem (62). Then by setting $v_n := |x|^\gamma u_n$, it follows that v_n satisfies

$$(65) \quad L_\gamma(v_n) = |x|^{-\gamma} \frac{h_n(x)}{(|x|^{-\gamma} v_n + \frac{1}{n})^\sigma} \leq |x|^{-(1-\sigma)\gamma} \frac{h_n}{v_n^\sigma} \quad \text{in } \Omega,$$

where L_γ was defined in (10). Using v_n^σ as a test function in (65) we obtain that

$$\frac{a_{N,s}}{2} \frac{4\sigma}{(\sigma+1)^2} \iint_{D_\Omega} \frac{(v_n^{\frac{\sigma+1}{2}}(x) - v_n^{\frac{\sigma+1}{2}}(y))^2}{|x-y|^{N+2s}} \frac{dx}{|x|^\gamma} \frac{dy}{|y|^\gamma} \leq \int_\Omega \frac{h_n}{|x|^{(1-\sigma)\gamma}} dx \leq C,$$

with C independent of n .

Thus we conclude that the sequence $\{v_n^{\frac{\sigma+1}{2}}\}_{n \in \mathbb{N}}$ is bounded in the weighted Sobolev space $Y_0^{s,\gamma}(\Omega)$ and hence there exists u_0 such that, up to a subsequence,

$$v_n^{\frac{\sigma+1}{2}} \rightharpoonup v_0^{\frac{\sigma+1}{2}} \quad \text{weakly in } Y_0^{s,\gamma}(\Omega).$$

Since $L_\gamma(v_n) \geq 0$, then using the fact that $\frac{\sigma+1}{2} < 1$, also $L_\gamma(v_n^{\frac{\sigma+1}{2}}) \geq 0$. Hence by the monotonicity of v_n and Lemma 2.18, we obtain that

$$v_n^{\frac{\sigma+1}{2}} \rightarrow v_0^{\frac{\sigma+1}{2}} \quad \text{strongly in } Y_0^{s,\gamma}(\Omega).$$

Passing to the limit in (65), it follows that v_0 solves

$$\begin{cases} L_\gamma(v_0) &= |x|^{-(1-\sigma)\gamma} \frac{h}{v_0^\sigma} & \text{in } \Omega, \\ v_0 &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

in the weak sense. Defining $u_0 := |x|^{-\gamma} v_0$, then $\lambda \frac{u_0}{|x|^{2s}} \in L^1(\Omega)$ and u_0 is a weak solution of problem (61). \square

To end this section, we consider the problem in the whole space, that is, $\Omega = \mathbb{R}^N$. Then we will work in the space $\dot{H}^s(\mathbb{R}^N)$ defined the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the Gagliardo seminorm

$$[\phi]_{\dot{H}^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

We obtain the following existence result.

Theorem 5.4. *Consider the problem*

$$(66) \quad \begin{cases} (-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + \frac{h(x)}{u^\sigma} & \text{in } \mathbb{R}^N, \\ u &> 0 & \text{in } \mathbb{R}^N. \end{cases}$$

Then

- (i) If $\sigma = 1$, then for all $h \in L^1(\mathbb{R}^N)$, problem (66) has a solution $u \in \dot{H}^s(\mathbb{R}^N)$.
- (ii) If $\sigma > 1$, then for all $h \in L^1(\mathbb{R}^N)$, problem (66) has a weak solution u such that $G_k(u) \in \dot{H}^s(\mathbb{R}^N)$ and $T_k^{\frac{\sigma+1}{2}}(u) \in \dot{H}^s(\mathbb{R}^N)$, for all $k > 0$. Moreover, if

$$(67) \quad \frac{4\sigma}{(\sigma+1)^2} - \frac{\lambda}{\Lambda_{N,s}} > 0,$$

then $u^{\frac{\sigma+1}{2}} \in \dot{H}^s(\mathbb{R}^N)$.

- (iii) If $\sigma < 1$ and $h \in L^m(\mathbb{R}^N)$ with $m = (\frac{2_s^*}{1-\sigma})'$, then problem (66) has a solution u such that $u \in \dot{H}^s(\mathbb{R}^N)$.
- (iv) If $\sigma < 1$ and (67) holds, then for all $h \in L^1(\mathbb{R}^N)$ problem (66) has a weak solution u such that $u^{\frac{\sigma+1}{2}} \in \dot{H}^s(\mathbb{R}^N)$.

Proof. Consider u_n to be the unique positive solution to the approximated problem

$$(68) \quad \begin{cases} (-\Delta)^s u_n &= \lambda \frac{u_n}{|x|^{2s} + \frac{1}{n}} + \frac{h_n(x)}{(u_n + \frac{1}{n})^\sigma} & \text{in } B_n(0), \\ u_n &> 0 & \text{in } B_n(0), \\ u_n &= 0 & \text{in } \mathbb{R}^N \setminus B_n(0). \end{cases}$$

It is clear that u_n is increasing with n .

If $\sigma = 1$, taking u_n as a test function in (68) and using the Hardy-Sobolev inequality it follows that

$$\frac{a_{N,s}}{2} \left(1 - \frac{\lambda}{\Lambda_{N,s}}\right) [u_n]_{\dot{H}^s(\mathbb{R}^N)}^2 \leq C.$$

Hence $\{u_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $\dot{H}^s(\mathbb{R}^N)$ and then, up to a subsequence, $u_n \rightharpoonup u$ weakly in $X^s(\mathbb{R}^N)$, where u solves (66). Using the monotonicity of u_n and a straightforward adaptation of Lemma 2.18 we can prove that $u_n \rightarrow u$ strongly in $\dot{H}^s(\mathbb{R}^N)$, which proves (i), and (iii) similarly follows.

To prove (ii) we take $G_k(u_n)$ as a test function in (68) and performing the same computations as in the proof of Theorem 5.2 we conclude.

Finally, (iv) follows closely using the arguments in the proof of Theorem 5.2. In particular, the existence of $u \in L^1(\Omega)$ is a consequence of the uniform bounds of $\{G_k(u_n)\}_{n \in \mathbb{N}}$ and $\{T_k^{\frac{\sigma+1}{2}}(u_n)\}_{n \in \mathbb{N}}$ in $\dot{H}^s(\mathbb{R}^N)$ and the fact that $\frac{\sigma+1}{2} < 1$. \square

Remark 5.5. In a similar way to the results in [11] the problem

$$(69) \quad \begin{cases} (-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + \mu \frac{h(x)}{u^\sigma} + u^p & \text{in } \Omega, \\ u &> 0 & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

can be analyzed. In fact the existence of a minimal solution for $p < p_+(\lambda)$ and $\lambda \leq \Lambda_{N,s}$ and μ small can be done with minor analytical changes. Indeed,

Theorem 5.6. Assume that in problem (69) one of the following conditions holds:

- (i) $\sigma < 1$, $p < 2_s^* - 1$ and $h \in L^\theta(\Omega)$ for some $\theta \geq \frac{2_s^*}{2_s^* - (1-\sigma)}$.
- (ii) $\sigma \geq 1$, $p < 2_s^* - 1$ and $h \in L^{\frac{2_s^*}{2_s^* - (1-\sigma_1)}}(\Omega)$ for some $\sigma_1 < 1$.
- (iii) $\sigma > 0$, $2_s^* - 1 \leq p < p_+(\lambda)$ and $h \in L^\infty(\Omega)$.

Then there exists $\mu^* > 0$ such that for all $\mu < \mu^*$, problem (69) has a minimal weak solution u and, moreover, for all $\mu > \mu^*$, problem (69) has no positive solution.

The existence of a second positive solution in the cases (i) and (ii) for μ small enough is easy to obtain. The result for all $\mu < \mu^*$ by a direct method seems to be an open problem.

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